POST-GRADUATE DEGREE PROGRAMME (CBCS)

M.SC. IN MATHEMATICS

SEMESTER-III

PAPER : DSE 3.3
(Applied Stream)

Modelling of Biological Systems

Dynamical Systems

Self-Learning Material

DIRECTORATE OF OPEN AND DISTANCE LEARNING
UNIVERSITY OF KALYANI
Kalyani, Nadia
West Bengal

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Director's Message

Satisfying the varied needs of distance learners, overcoming the obstacle of Distance and reaching the unreached students are the three fold functions catered by Open and Distance Learning (ODL) systems. The onus lies on writers, editors, production professionals and other personnel involved in the process to overcome the challenges inherent to curriculum design and production of relevant Self Learning Materials (SLMs). At the University of Kalyani a dedicated team under the able guidance of the Hon'ble Vice-Chancellorhas invested its best efforts, professionally and in keeping with the demands of Post Graduate CBCS Programmes in Distance Mode to devise a self-sufficient curriculum for each course offered by the Directorate of Open and Distance Learning (DODL), University of Kalyani.

Development of printed SLMs for students admitted to the DODL within a limited time to cater to the academic requirements of the Course as per standards set by Distance Education Bureau of the University Grants Commission, New Delhi, India under Open and Distance Mode UGC Regulations, 2020 had been our endeavor. We are happy to have achieved our goal.

Utmost care and precision have been ensured in the development of the SLMs, making them useful to the learners, besides avoiding errors as far as practicable. Further suggestions from the stakeholders in this would be welcome.

During the production-process of the SLMs, the team continuously received positive stimulations and feedback from Professor **(Dr.) Amalendu Bhunia, Hon'ble Vice-Chancellor, University of Kalyani**, who kindly accorded directions, encouragements and suggestions, offered constructive criticism to develop it with in proper requirements. We gracefully, acknowledge his inspiration and guidance.

Sincere gratitude is due to the respective chairpersons as well as each and every member of PGBOS (DODL), University of Kalyani. Heartfelt thanks are also due to the Course Writers-faculty members at the DODL, subject-experts serving at University Post Graduate departments and also to the authors and academicians whose academic contributions have enriched the SLMs. We humbly acknowledge their valuable academic contributions. I wouldespecially like to convey gratitude to all other University dignitaries and personnel involved either at the conceptual or operational level of the DODL of University of Kalyani.

Their persistent and coordinated efforts have resulted in the compilation of comprehensive, learner-friendly, flexible texts that meet the curriculum requirements of the Post Graduate Programme through Distance Mode.

All the Self Learning Materials are self writing and collected from e-book, journals and websites.

Professor (Dr.) Sanjib Kumar Datta
Director
Directorate of Open and Distance Learning
University of Kalyani

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Discipline Specific Elective Paper

APPLIED STREAM

DSE 3.3
Marks : 100 (SEE : 80; IA : 20); Credit : 6

Modelling of Biological Systems (Marks : 50 (SEE: 40; IA: 10))
Dynamical Systems (Marks : 50 (SEE: 40; IA: 10))

Syllabus

Block I

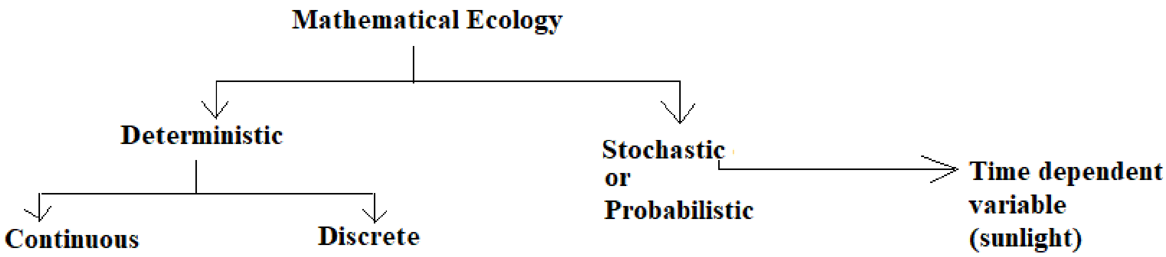
- **Unit 1:** Mathematical models in ecology: Discrete and Continuous population models for single species. Logistic models and their stability analysis. Stochastic birth and death processes.
- **Unit 2:** Continuous models for two interacting populations: Lotka-Volterra model of predator -prey system, Kolmogorov model. Trophic function. Gauss's Model.
- **Unit 3:** Leslie-Gower predator-prey model. Analysis of predator-prey model with limit cycle behavior, parameter domains of stability. Nonlinear oscillations in predator-prey system.
- **Unit 4:** Deterministic Epidemic Models: Deterministic model of simple epidemic, Infection through vertical and horizontal transmission, General epidemic- Kermack-Mckendrick Threshold Theorem.
- **Unit 5:** Delay Models: Discrete and Distributed delay models. Stability of population steady states.
- **Unit 6:** Spatial Models: Formulating spatially structured models. Spatial steady states: Linear and nonlinear problems. Models of spread of population.
- **Unit 7:** Blood flow models: Basic concepts of blood flow and its special characteristics. Application of Poiseulle's law to the study of bifurcation in an artery.
- **Unit 8:** Pulsatile flow of blood in rigid and elastic tubes. Aortic diastolic-systolic pressure waveforms. Moen-Korteweg expression for pulse wave velocity in elastic tube. Blood flow through artery with mild stenosis.
- **Unit 9:** Models for other fluids: Peristaltic motion in a channel and in a tube. Two dimensional flow in renal tubule. Lubrication of human joints.
- **Unit 10:** Models in Pharmacokinetics: Compartments, Basic equations, single and two compartment models.

Block II

- **Unit 11:** Autonomous and non-autonomous systems: Orbit of a map, fixed point, equilibrium point, periodic point, circular map, configuration space and phase space.
- **Unit 12:** Nonlinear oscillators-conservative system. Hamiltonian system. Various types of oscillators in nonlinear system viz. simple pendulum, and rotating pendulum.
- **Unit 13:** Limit cycles: Poincaré-Bendixon theorem (statement only). Criterion for the existence of limit cycle for Liénard's equation.
- **Unit 14:** Stability: Definition in Liapunov sense. Routh-Hurwitz criterion for nonlinear systems.
- **Unit 15:** Liapunov's criterion for stability. Stability of periodic solutions. Floquet's theorem.
- **Unit 16:** Solutions of nonlinear differential equations by perturbation method: Secular term. Nonlinear damping.
- **Unit 17:** Solutions for the equations of motion of a simple pendulum, Duffing and Vanderpol oscillators.
- **Unit 18:** Bifurcation Theory: Origin of Bifurcation, Bifurcation Value, Normalisation, Resonance, Stability of a fixed point.
- **Unit 19:** Bifurcation of equilibrium solutions – the saddle node bifurcation, the pitch-fork bifurcation, Hopf-bifurcation.
- **Unit 20:** Randomness of orbits of a dynamical system: The Lorentz equations, Chaos, Strange attractors.

Contents

Unit 1



1.1 Single Species population growth

1.1.1 Malthusian Growth(Malthus)

Let $N(t)$ be the concentration of the population at the time t. $\frac{dN}{dt}$ is the growth rate of that population.

$$\begin{aligned}
\frac{\frac{dN}{dt}}{N} &= \text{Growth rate per unit of concentration} \\
&= b - d \\
&= \text{birth rate} - \text{death rate} \\
&= r(\text{constant}) \\
\Rightarrow \frac{dN}{dt} &= rN \\
\Rightarrow \frac{dN}{N} &= rdt \\
\Rightarrow \log N &= rt + \log A \\
\Rightarrow N(t) &= Ae^{rt}
\end{aligned}$$

Initially, $t = 0$, $N(t) = N_0$. Hence, $N_0 = A$.

Therefore,

$$\boxed{N(t) = N_0 e^{rt}}$$

This is Malthusian Growth equation.

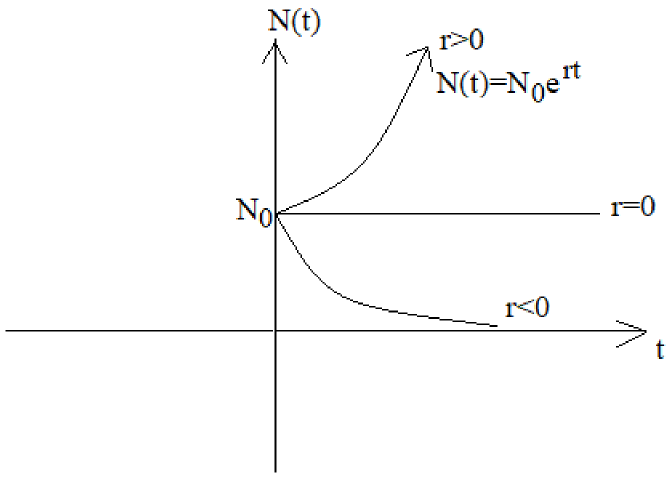


Figure 1.1.1

1.1.2 Logistic Population Growth Model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = rN - \frac{rN^2}{K}$$

$N(t) \implies$ population size at any time t.

$r \implies$ growth rate.

$K \implies$ carrying capacity of the population.

$$\Rightarrow \frac{dN}{dt} = rN - \frac{rN^2}{K}$$

$$\Rightarrow \frac{dN}{rN - \frac{rN^2}{K}} = dt$$

$$\Rightarrow \frac{dN}{\frac{r}{K}(KN - N^2)} = dt$$

$$\Rightarrow \frac{K}{r} \cdot \frac{dN}{N(K-N)} = dt$$

$$\Rightarrow \frac{K}{K \times r} \int \frac{(K - N + N)}{N(K-N)} dN = \int dt$$

$$\Rightarrow \frac{K}{r} \times \frac{1}{K} \int \frac{dN}{N} + \frac{K}{r} \times \frac{1}{K} \int \frac{dN}{K-N} = \int dt$$

$$\Rightarrow \frac{1}{r} \log N - \frac{1}{r} \log(K - N) = t + \log A_1$$

$$\Rightarrow \frac{1}{r} \log \frac{N}{K-N} = t + \log A_1$$

$$\Rightarrow \log \frac{N}{K-N} = rt + \log A$$

$$\Rightarrow \frac{N}{A(K-N)} = e^{rt}$$

$$
\begin{aligned}
&\Rightarrow \quad \frac{A(K-N)}{N} = e^{-rt} \\
&\Rightarrow \quad \left(\frac{K}{N} - 1\right) = \frac{1}{A}e^{-rt} \\
&\Rightarrow \quad \frac{K}{N} = 1 + \frac{1}{A}e^{-rt} \\
&\Rightarrow \quad \frac{N}{K} = \frac{A}{A + e^{-rt}}
\end{aligned}
$$

Initially, $t = 0, N = N_0$.

$$
\begin{aligned}
& \quad N_0 = \frac{AK}{A+1} \\
&\Rightarrow \quad AN_0 + N_0 = AK \\
&\Rightarrow \quad A(N_0 - K) = -N_0 \\
&\Rightarrow \quad A = \frac{N_0}{K - N_0}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
N(t) &= \frac{\frac{N_0 K}{K-N_0}}{\frac{N_0}{K-N_0} + e^{-rt}} \\
&= \frac{N_0 K}{N_0 + e^{-rt}(K - N_0)}
\end{aligned}
$$

$\underline{r > 0}$: As $t \to \infty$, $N(t) \to K$
$\underline{r < 0}$: As $t \to \infty$, $N(t) \to 0$.

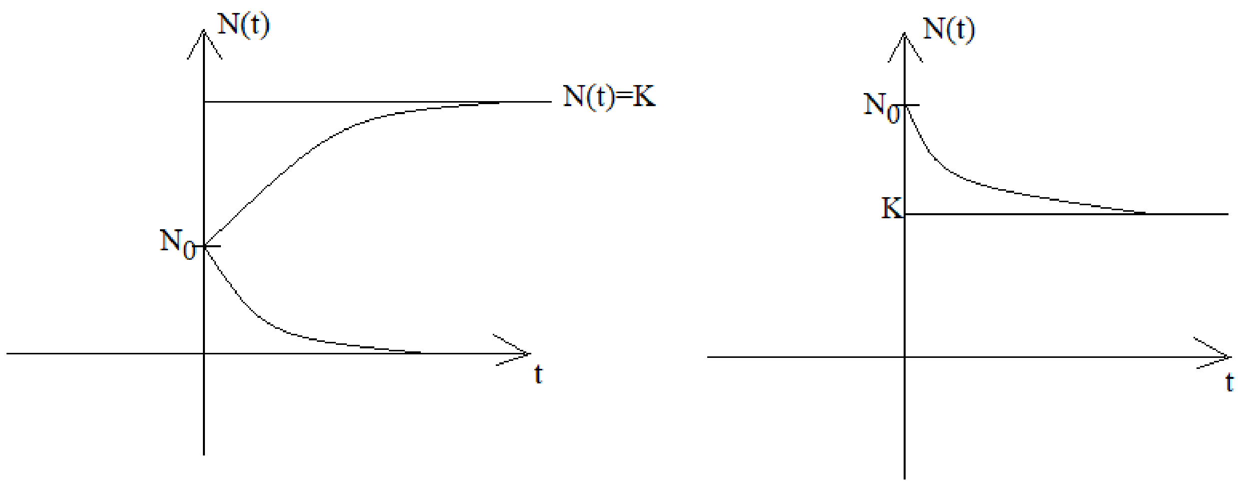


Figure 1.1.2

For

$$\begin{aligned}
&\frac{dN}{dt} = rN - \frac{rN^2}{K} = 0 \\
\Rightarrow\quad & N\left(r - \frac{rN}{K}\right) = 0 \\
& N = 0r = \frac{rN}{K} \\
& \implies N = K
\end{aligned}$$

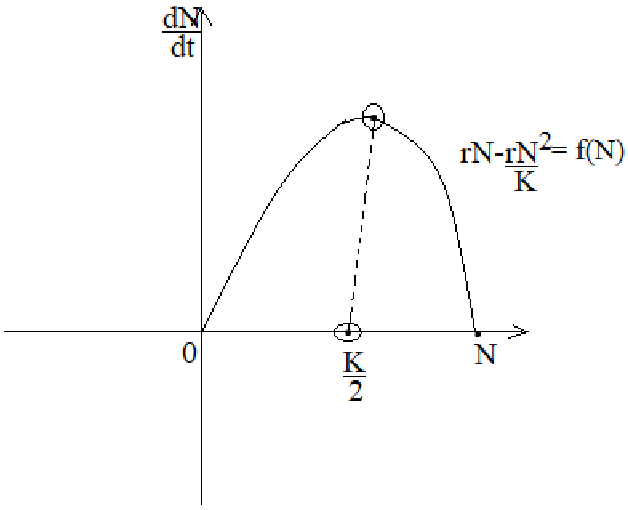


Figure 1.1.3

Logistic model is also called Verlhust Model.

1.1.3 Gompertz Growth Model

$$\begin{aligned}
&\frac{1}{N}\frac{dN}{dt} = r - S\log N \\
\Rightarrow\quad &\frac{1}{N}\frac{dN}{dt} = ae^{-SN}, \quad \left[\frac{r}{S} = K\right]
\end{aligned}$$

For equilibrium,

$$\begin{aligned}
\frac{dN}{dt} = 0 \;&\Rightarrow\; r - S\log N = 0 \\
&\Rightarrow\; \log N = \frac{r}{S} \\
&\Rightarrow\; N = e^{\frac{r}{S}}
\end{aligned}$$

Let us substitute $X = \log N$.

Then,

$$
\begin{aligned}
& \frac{dX}{dt} = \frac{1}{N}\frac{dN}{dt} = r - SX \\
\Rightarrow\quad & \frac{dX}{S\left(\frac{r}{S} - X\right)} = dt \\
\Rightarrow\quad & -\frac{1}{S}\log\left(\frac{r}{S} - X\right) = t + \frac{1}{S}\log A_1 \\
\Rightarrow\quad & -\frac{1}{S}\log(r - SX) = t + \frac{1}{S}\log A \\
\Rightarrow\quad & \log[A^{\frac{1}{S}}(r - SX)^{\frac{1}{S}}] = -t \\
\Rightarrow\quad & A^{\frac{1}{S}}(r - SX)^{\frac{1}{S}} = e^{-t} \\
\Rightarrow\quad & (r - SX)^{\frac{1}{S}} = \frac{1}{A^{\frac{1}{S}}}e^{-t} \\
\Rightarrow\quad & A(r - SX) = e^{-St} \\
\Rightarrow\quad & (r - SX) = \frac{1}{A}e^{-St} \\
\Rightarrow\quad & X = \frac{r}{S} - \frac{1}{SA}e^{-St} \\
\Rightarrow\quad & \log N = \frac{1}{S}\left[r - \frac{1}{A}e^{-St}\right] \\
\Rightarrow\quad & N = \exp\frac{1}{S}(r - \frac{1}{A}e^{-St})
\end{aligned}
$$

As $t \to \infty$, $N \to e^{r/S}$.

1.1.4 Discrete type logistic Growth Model

$$N(t) = N = \frac{ke^{r(t-t_0)}}{1 + e^{r(t-t_0)}} = N_t \tag{1.1.1}$$

We know that

$$N(t) = \frac{N_0 k}{N_0 + e^{-rt(k-N_0)}} \quad \text{[when } t = 0, N = N_0\text{]}$$

Now,

$$N(t) = \frac{Ak}{A + e^{-rt}} = \frac{Ake^{rt}}{Ae^{rt} + 1}$$

When $t = t_0$, then $N = \frac{k}{2}$, so

$$
\begin{aligned}
& \frac{k}{2} = \frac{Ake^{rt_0}}{Ae^{rt_0} + 1} \\
\Longrightarrow\quad & \frac{1}{2} = \frac{Ae^{rt_0}}{Ae^{rt_0} + 1} \\
\Longrightarrow\quad & Ae^{rt_0} + 1 = 2Ae^{rt_0} \\
\Longrightarrow\quad & Ae^{rt_0} = 1 \\
\Longrightarrow\quad & A = \frac{1}{e^{rt_0}} = e^{-rt_0}
\end{aligned}
$$

Then,

$$N(t) = \frac{k \cdot e^{-rt_0} \cdot e^{rt}}{1 + e^{-rt_0} \cdot e^{rt}} = \frac{k \cdot e^{r(t-t_0)}}{1 + e^{r(t-t_0)}}$$

Now, at $t = t + 1$,

$$N_{t+1} = N(t+1) = \frac{ke^{r(t+1-t_0)}}{1 + e^{r(t+1-t_0)}} \tag{1.1.2}$$

From (1.1.1),

$$\begin{aligned} & N_t + N_t e^{r(t-t_0)} = ke^{r(t-t_0)} \\ \implies & N_t = e^{r(t-t_0)}(k - N_t) \\ \implies & e^{r(t-t_0)} = \frac{N_t}{k - N_t} \end{aligned}$$

Now, from (1.1.2)

$$\begin{aligned} N_{t+1} &= \frac{ke^r e^{r(t-t_0)}}{1 + e^r e^{r(t-t_0)}} \\ &= \frac{ke^r \left(\frac{N_t}{k-N_t}\right)}{1 + e^r \left(\frac{N_t}{k-N_t}\right)} \\ &= \frac{ke^r N_t}{k - N_t + e^r N_t} \\ &= \frac{kN_t e^r}{k + N_t(e^r - 1)} \end{aligned}$$

Therefore,

$$\boxed{N_{t+1} = \frac{kN_t e^r}{k + N_t(e^r - 1)}}$$

$$N_{t+2} = \frac{kN_{t+1} e^r}{k + N_{t+1}(e^r - 1)}$$

1.1.5 Discrete type Malthusian Model

$$\begin{aligned} N_t &= N(t) = N_0 e^{rt} \\ N_{t+1} &= N_0 e^{r(t+1)} = N_0 e^r e^{rt} \end{aligned}$$

$$\boxed{N_{t+1} = e^r N_t}$$

1.2 Two species population growth

$+ -$	Predator-prey Model
$- +$	Host-Pathogen Model
$- -$	Competition
$+ +$	Symbiosis

1.2.1 Lotka-Volterra predator-prey Model (Host-Pathogen Model)

Let $H(t)$ be the concentration of the prey (host) population at time t and $P(t)$ be the concentration of predator (parasite) population, then the governing equation is

$$\left.\begin{array}{lll}\text{(Host/Prey)} & \frac{1}{H}\frac{dH}{dt} & = a_1 - b_1 P \\ \text{(Predator/Parasite)} & \frac{1}{P}\frac{dP}{dt} & = -a_2 + b_2 H\end{array}\right\} \tag{1.2.1}$$

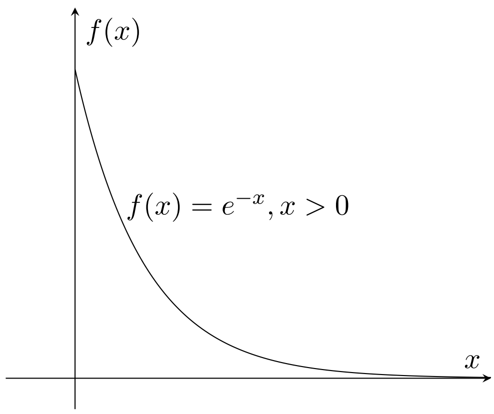
where a_1, a_2, b_1, b_2 are all positive.
$a_1 \to$ growth rate pf prey in absence of predator,
$b_1 \to$ predation rate,
$a_2 \to$ death rate of predator in absence of prey,
$b_2 \to$ conversion rate

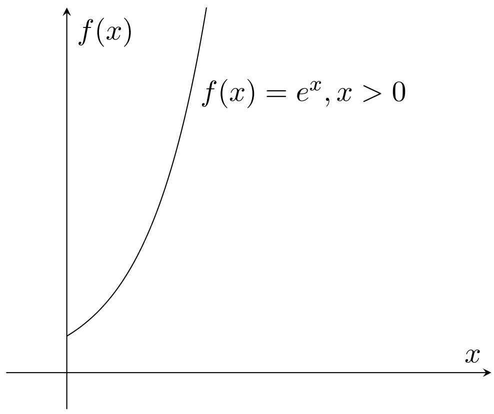
$$\begin{aligned} & \frac{P}{H}\frac{dH}{dP} = \frac{a_1 - b_1 P}{-a_2 + b_2 H} \\ \implies & \frac{-a_2 + b_2 H}{H} dH = \frac{a_1 - b_1 P}{P} dP \\ \implies & -a_2 \log H + b_2 H = a_1 \log P - b_1 P + c \\ \implies & b_1 P + b_2 H - a_2 \log H - a_1 \log P = c \end{aligned}$$

Initially, $t = 0, H = H_0, P = P_0$. Therefore

$$c = b_1 P_0 + b_2 H_0 - a_2 \log H_0 - a_1 \log P_0$$

$$\therefore \boxed{b_1(P - P_0) + b_2(H - H_0) - a_2 \log \frac{H}{H_0} - a_1 \log \frac{P}{P_0} = 0}$$





1.3 Probabilistic Model (Stochastic)

1.3.1 Simple Birth Model

Let in the time interval $t \to t + \Delta t$ (Δt is so very small) $(t, t + \Delta t)$, the probability of one birth is proportional to Δt, i.e. $\lambda \Delta t$, λ is constant. Let $P_N(t)$ be the probability that at time t, the number of individual is N. Then

$$P_N(t + \Delta t) = P_N(t)[1 - \lambda N \Delta t] + P_{N-1}(t)\lambda(N-1)\Delta t + O(t)$$

Then

$$\lim_{\Delta t\to 0}\frac{P_N(t+\Delta t)-P_N(t)}{\Delta t} = -\lambda N P_N(t) + \lambda(N-1)P_{N-1}(t) + \lim_{\Delta t\to 0}\frac{O(\Delta t)}{\Delta t}$$

$$\therefore \frac{d}{dt}(P_N(t)) = -\lambda N P_N(t) + \lambda(N-1)P_{N-1}(t) \tag{1.3.1}$$

This is called **differential difference equation**.
The initial conditions are $P_i(0) = 1, P_j(0) = 0\ (i \neq j)$. Putting $N = i$ in (1.3.1), we get

$$\frac{d}{dt}P_i(t) = -\lambda i P_i(t) + \lambda(i-1)\times 0 = -\lambda i P_i(t)$$

Solving the above equation, we get

$$\boxed{P_i(t) = Ae^{-\lambda_i t}}$$

Initially at $t = 0, P_i(0) = 1$, so $A = 1$ and hence

$$\therefore P_i(t) = e^{-\lambda_i t}$$

Putting $N = i+1$ in (1.3.1), we get

$$\begin{aligned} &\frac{d}{dt}(P_{i+1}(t)) = -\lambda(i+1)P_{i+1}(t) + \lambda i P_i(t) \\ \implies\ &\frac{d}{dt}(P_{i+1}(t)) = -\lambda(i+1)P_{i+1}(t) + \lambda i e^{-\lambda_i t} \\ \implies\ &\frac{d}{dt}(P_{i+1}(t)) + \lambda(i+1)P_{i+1}(t) = \lambda i e^{-\lambda_i t} \end{aligned}$$

This is a linear equation and its IF (integrating factor) is $e^{\lambda(i+1)t}$.

$$\begin{aligned} P_{i+1}(t)e^{\lambda(i+1)t} &= \int \lambda i e^{-\lambda i t} e^{\lambda(i+1)t}dt + c \\ &= \int \lambda i e^{\lambda t}dt + c \\ &= \lambda i \frac{e^{\lambda t}}{\lambda} + c \end{aligned}$$

So,

$$P_{i+1}(t)e^{\lambda(i+1)t} = ie^{\lambda t} + B$$

Initially when $t = 0, P_{i+1}(0) = 0$, this implies $B + i = 0$ and so $B = -i$. Therefore,

$$\begin{aligned} &P_{i+1}(t)e^{\lambda(i+1)t} = ie^{\lambda t} - i \\ \implies\ &P_{i+1}(t) = ie^{-\lambda(i+1)t}(e^{\lambda t} - 1) \\ \implies\ &P_{i+1}(t) = ie^{-\lambda i t}(1 - e^{-\lambda t}) \end{aligned}$$

Now,

$$\boxed{P_i(t) = e^{-\lambda_i t}}$$

$$\boxed{P_{i+1}(t) = ie^{-\lambda i t}(1 - e^{-\lambda t})}$$

Proceeding in this way,

$$\begin{aligned} P_{i+2}(t) &= \frac{i(i+1)}{2} e^{-\lambda i t}[1-e^{-\lambda t}]^2 \\ &= \binom{i+1}{2} e^{-\lambda i t}[1-e^{-\lambda t}]^2 \\ &\vdots \end{aligned}$$

Therefore,

$$\boxed{P_N(t) = \binom{N-1}{N-i} e^{-\lambda i t}[1-e^{-\lambda t}]^{N-i}}$$

1.3.2 Pure Death Model

Let the probability of one death per individual in time interval $(t, t+\Delta t)$ be proportional to Δt, i.e. $\mu \Delta t$, where μ is a constant.

Then

$$\begin{aligned} & P_N(t+\Delta t) = P_N(t)[1-\mu N \Delta t] + P_{N+1}(t)\mu(N+1)\Delta t + O(\Delta t) \\ \Longrightarrow \quad & \lim_{\Delta t \to 0} \frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} = -\mu N P_N(t) + \mu(N+1)P_{N+1}(t) + \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} \\ \Longrightarrow \quad & \frac{d}{dt}(P_N(t)) = -\mu N P_N(t) + \mu(N+1)P_{N+1}(t) \end{aligned} \tag{1.3.2}$$

Initially at $t = 0$, $P_i(0) = 1$ and $P_j(0) = 0$ for $j > i$. Putting $N = i, i-1, \ldots$ in (1.3.2), we get

$$\begin{aligned} & \frac{d}{dt}(P_i(t)) = -\mu i P_i(t) + \mu(i+1)P_{i+1}(t) \\ \Longrightarrow \quad & \frac{d}{dt}(P_i(t)) = -\mu i P_i(t) \qquad [\text{since } P_{i+1}(t) = 0] \end{aligned}$$

Solving the above equation, we get

$$P_i(t) = A e^{-\mu i t}$$

As at $t = 0$, $P_i(0) = 1$, so $A = 1$ and therefore

$$\boxed{P_i(t) = e^{-\mu i t}}$$

Putting $N = i - 1$ in (1.3.2), we obtain

$$\begin{aligned} & \frac{d}{dt}P_{i-1}(t) = -\mu(i-1)P_{i-1}(t) + \mu(i-1+1)P_i(t) \\ \Longrightarrow \quad & \frac{d}{dt}P_{i-1}(t) = -\mu(i-1)P_{i-1}(t) + \mu i e^{-\mu i t} \\ \Longrightarrow \quad & \frac{d}{dt}P_{i-1}(t) + \mu(i-1)P_{i-1}(t) = \mu i e^{-\mu i t} \end{aligned}$$

This is a linear equation and its I.F. is $e^{\mu(i-1)t}$. Therefore,

$$\begin{aligned} P_{i-1}(t)e^{\mu(i-1)t} &= \int \mu i e^{-\mu i t} e^{\mu(i-1)t} dt + c \\ &= \int \mu i e^{-\mu t} dt + c \\ &= -\frac{\mu i e^{-\mu t}}{\mu} + B \\ &= -ie^{-\mu t} + B \end{aligned}$$

Initially at $t = 0, P_{i-1} = 0$, so $B - i = 0$, i.e., $B = i$. Hence,

$$\begin{aligned} & P_{i-1}(t)e^{\mu(i-1)t} = -ie^{-\mu t} + i = i(1 - e^{-\mu t}) \\ \implies \quad & P_{i-1}(t) = ie^{-\mu(i-1)t}(1 - e^{-\mu t}) \\ \implies \quad & P_{i-1}(t) = ie^{-\mu i t}(e^{\mu t} - 1) \end{aligned}$$

Therefore,

$$\begin{aligned} P_i(t) &= e^{-\mu i t} \\ P_{i-1}(t) &= ie^{-\mu i t}(e^{\mu t} - 1) \end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned} P_{i-2}(t) &= \binom{i}{2} e^{-\mu i t}(e^{\mu t} - 1)^2 \\ P_{i-3}(t) &= \binom{i}{3} e^{-\mu i t}(e^{\mu t} - 1)^3 \\ &\vdots \\ P_N(t) &= \binom{i}{i-N} e^{-\mu i t}(e^{\mu t} - 1)^{i-N} \quad \text{for } N < i, \\ &= \binom{i}{N} e^{-\mu i t}(e^{\mu t} - 1)^{i-N} \qquad \left[\text{as } \binom{n}{r} = \binom{n}{n-r}\right] \end{aligned}$$

Therefore, for $N < i$

$$\boxed{P_N(t) = \binom{i}{N} e^{-\mu i t}(e^{\mu t} - 1)^{i-N}}$$

1.3.3 Simple Birth and Death Process

Let, in time interval $(t, t+\Delta t)$, the probability of one birth be proportional to Δt, i.e. $\lambda\Delta t$ and the probability of one death be also proportional to Δt, i.e. $\mu\Delta t$. Then

$$P_N(t + \Delta t) = P_{N-1}(t)\lambda(N-1)\Delta t + P_{N+1}(t)\mu(N+1)\Delta t + P_N(t)(1 - N\lambda\Delta t - \mu N\Delta t) + O(\Delta t)$$

$$\begin{aligned} \implies \quad & \lim_{\Delta t \to 0} \frac{P_N(t + \Delta t) - P_N(t)}{\Delta t} = \lambda(N-1)P_{N-1}(t) + \mu(N+1)P_{N+1}(t) - (\lambda + \mu)NP_N(t) + 0 \\ \implies \quad & \frac{d}{dt}(P_N(t)) = \lambda(N-1)P_{N-1}(t) + \mu(N+1)P_{N+1}(t) - (\lambda + \mu)NP_N(t) \end{aligned} \tag{1.3.3}$$

Multiplying (1.3.3) by N and summing over N, we get

$$\sum_{N=0}^{\infty} N\frac{d}{dt}(P_N(t)) = \lambda\sum_{N=1}^{\infty} N(N-1)P_{N-1}(t) + \mu\sum_{N=0}^{\infty} N(N+1)P_{N+1}(t) - (\lambda+\mu)\sum_{N=0}^{\infty} N^2 P_N(t)$$

Rewriting this using dummy indices, we obtain

$$\begin{aligned}
\frac{d}{dt}\sum_{N=0}^{\infty} NP_N(t) &= \lambda\sum_{N=0}^{\infty}(N+1)NP_N(t) + \mu\sum_{N=0}^{\infty}(N-1)NP_N(t) - (\lambda+\mu)\sum_{N=0}^{\infty} N^2P_N(t) \\
&= \sum_{N=0}^{\infty} P_N(t)[N(N+1)\lambda + N(N-1)\mu - (\lambda+\mu)N^2] \\
&= \sum_{N=0}^{\infty} P_N(t)[N^2\lambda + N\lambda + N^2\mu - N\mu - \lambda N^2 - \mu N^2] \\
&= \sum_{N=0}^{\infty} N(\lambda-\mu)P_N(t) \\
&= (\lambda-\mu)\sum_{N=0}^{\infty} NP_N(t)
\end{aligned}$$

Taking $M(N,t) = \sum_{N=0}^{\infty} NP_N(t)$, we have

$$\frac{d}{dt}M(N,t) = (\lambda-\mu)M(N,t)$$

Solving this we get

$$M(N,t) = Ae^{(\lambda-\mu)t}$$

Initially, at $t = 0, N = i$ and $P_i(0) = 1, P_j(0) = 0$ for $j \neq i$. So

$$M(N,0) = P_1(0) + 2P_2(0) + \cdots + iP_i(0) + \cdots = i$$

and hence $A = i$.

$$\therefore \boxed{M(N,t) = ie^{(\lambda-\mu)t}}$$

Multiplying (1.3.3) by N^2 and summing over N we get

$$\begin{aligned}
\sum_{N=0}^{\infty} N^2\frac{d}{dt}(P_N(t)) &= \lambda\sum_{N=1}^{\infty} N^2(N-1)P_{N-1}(t) + \mu\sum_{N=0}^{\infty} N^2(N+1)P_{N+1}(t) - (\lambda+\mu)\sum_{N=0}^{\infty} N^3P_N(t) \\
&= \lambda\sum_{N=0}^{\infty}(N+1)^2NP_N(t) + \mu\sum_{N=0}^{\infty}(N-1)^2NP_N(t) - (\lambda+\mu)\sum_{N=0}^{\infty} N^3P_N(t) \\
&= \sum_{N=0}^{\infty} NP_N(t)[\lambda N^2 + 2N\lambda + \lambda + \mu N^2 - 2N\mu + \mu - \lambda N^2 - \mu N^2] \\
&= \sum_{N=0}^{\infty} NP_N(t)[(\lambda-\mu)2N + (\lambda+\mu)] \\
&= 2(\lambda-\mu)\sum_{N=0}^{\infty} N^2P_N(t) + (\lambda+\mu)\sum_{N=0}^{\infty} NP_N(t)
\end{aligned}$$

Taking $M_2(N,t) = \sum_{N=0}^{\infty} N^2 P_N(t)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}M_2(N,t) = 2(\lambda-\mu)M_2(N,t) + (\lambda+\mu)M(N,t) \\
\implies \quad &\frac{d}{dt}M_2(N,t) - 2(\lambda-\mu)M_2(N,t) = (\lambda+\mu)ie^{(\lambda-\mu)t}
\end{aligned}$$

This is linear and its I.F. is $e^{-2(\lambda-\mu)t}$. Therefore,

$$\begin{aligned}
M_2(N,t)e^{-2(\lambda-\mu)t} &= i(\lambda+\mu)\int e^{(\lambda-\mu)t}e^{-2(\lambda-\mu)t}dt \\
&= i(\lambda+\mu)\frac{e^{-(\lambda-\mu)t}}{-(\lambda-\mu)} + C
\end{aligned}$$

$$\therefore M_2(N,t) = -\frac{i(\lambda+\mu)}{(\lambda-\mu)}e^{(\lambda-\mu)t} + Ce^{2(\lambda-\mu)t}$$

Initially at $t=0, N=i$ and $P_i(0)=1, P_j(0)=0$ for $j \neq i$. So

$$M_2(N,0) = M_2(i,0) = \sum_{N=0}^{\infty} N^2 P_N(t) = i^2$$

which in turn gives

$$i^2 = -\frac{i(\lambda+\mu)}{(\lambda-\mu)} + C \quad \implies \quad C = i^2 + \frac{i(\lambda+\mu)}{(\lambda-\mu)}$$

$$\therefore M_2(N,t) = -\frac{i(\lambda+\mu)}{(\lambda-\mu)}e^{(\lambda-\mu)t} + \left(i^2 + \frac{i(\lambda+\mu)}{(\lambda-\mu)}\right)e^{2(\lambda-\mu)t}$$

$$\begin{aligned}
\text{Var}(N,t) &= -\frac{i(\lambda+\mu)}{(\lambda-\mu)}e^{(\lambda-\mu)t} + \left(i^2 + \frac{i(\lambda+\mu)}{(\lambda-\mu)}\right)e^{2(\lambda-\mu)t} - i^2e^{2(\lambda-\mu)t} \\
&= \frac{i(\lambda+\mu)}{(\lambda-\mu)}e^{(\lambda-\mu)t}\left(e^{(\lambda-\mu)t} - 1\right)
\end{aligned}$$

Recall the equation (1.3.3)

$$\frac{d}{dt}(P_N(t)) = -(\lambda+\mu)NP_N(t) + \lambda(N-1)P_{N-1}(t) + \mu(N+1)P_{N+1}(t)$$

To solve this equation, we introduce a function known as generating function $\phi(z,t)$, defined as

$$\phi(z,t) = \sum_{N=0}^{\infty} z^N P_N(t) \tag{1.3.4}$$

Multiplying (1.3.3) by z^N and summing over N, we get

$$\sum_{N=0}^{\infty} z^N \frac{d}{dt}(P_N(t)) = -(\lambda+\mu)\sum_{N=0}^{\infty} z^N NP_N(t) + \lambda\sum_{N=0}^{\infty} z^N(N-1)P_{N-1}(t) + \mu\sum_{N=0}^{\infty} z^N(N+1)P_{N+1}(t) \tag{1.3.5}$$

From (1.3.4) we can derive that

$$\frac{\partial}{\partial t}\phi(z,t) = \sum_{N=0}^{\infty} z^N \frac{d}{dt}(P_N(t))$$
$$\frac{\partial}{\partial z}\phi(z,t) = \sum_{N=0}^{\infty} N z^{N-1} P_N(t)$$
$$\implies \quad z\frac{\partial}{\partial z}\phi(z,t) = \sum_{N=0}^{\infty} N z^{N} P_N(t)$$

Therefore, from (1.3.5), we get

$$\frac{\partial}{\partial t}\phi(z,t) = -(\lambda+\mu)z\frac{\partial}{\partial z}\phi(z,t) + \lambda z^2\frac{\partial}{\partial z}\phi(z,t) + \mu z\frac{\partial}{\partial z}\phi(z,t)$$
$$\implies \quad \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial z}[\lambda z^2 - \lambda z - \mu z + \mu]$$
$$\implies \quad \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial z}(\lambda z - \mu)(z-1) = 0$$

This is first order PDE. The auxiliary equations are

$$\frac{dt}{1} = -\frac{dz}{(z-1)(\lambda z-\mu)} = \frac{d\phi}{0}$$

Solving we get, $\phi = C_1$ (constant),

$$dt = -\frac{dz}{(z-1)(\lambda z-\mu)} = -\left[\frac{1}{z-1} - \frac{\lambda}{\lambda z-\mu}\right]\cdot\frac{dz}{\lambda-\mu}$$

Integrating both sides,

$$-(\lambda-\mu)t = \log(z-1) - \log(\lambda z-\mu) + \log A$$
$$\implies \quad -(\lambda-\mu)t = \log\frac{z-1}{\lambda z-\mu} + \log A$$
$$\implies \quad \frac{z-1}{\lambda z-\mu} = Be^{-(\lambda-\mu)t}$$
$$\implies \quad B = \frac{z-1}{\lambda z-\mu}e^{(\lambda-\mu)t}$$
$$\implies \quad C = \frac{1}{B} = \frac{\mu-\lambda z}{1-z}e^{-(\lambda-\mu)t}$$

The general solution is

$$\phi(z,t) = f\left(\frac{\mu-\lambda z}{1-z}e^{-(\lambda-\mu)t}\right) \tag{1.3.6}$$

where f is an arbitrary function.

Again

$$\phi(z,t) = \sum_{N=0}^{\infty} z^N P_N(t)$$

Putting $t=0, N=i$, we obtain

$$\phi(z,0) = z^i$$

Now from (1.3.6),

$$\phi(z,0) = f\left(\frac{\mu - \lambda z}{1-z}\right)$$

Thus $z^i = f(\xi)$ where

$$\begin{aligned} \xi &= \frac{\mu - \lambda z}{1-z} \\ \implies \quad \xi - \xi z &= \mu - \lambda z \\ \implies \quad z(\lambda - \xi) &= \mu - \xi \\ \implies \quad z &= \frac{\mu - \xi}{\lambda - \xi} = \frac{\xi - \mu}{\xi - \lambda} \end{aligned}$$

So we get

$$f(\xi) = \left(\frac{\xi - \mu}{\xi - \lambda}\right)^i$$

Therefore,

$$\begin{aligned} \phi(z,t) &= \left[\frac{\frac{\mu-\lambda z}{1-z} e^{-(\lambda-\mu)t} - \mu}{\frac{\mu-\lambda z}{1-z} e^{-(\lambda-\mu)t} - \lambda}\right]^i \\ &= \left[\frac{(\mu - \lambda z)e^{-(\lambda-\mu)t} - \mu(1-z)}{(\mu - \lambda z)e^{-(\lambda-\mu)t} - \lambda(1-z)}\right]^i \end{aligned}$$

Unit 2

2.1 Two species population growth

$+\,-$	Predator-prey Model
$-\,+$	Host-Pathogen Model
$-\,-$	Competition
$+\,+$	Symbiosis

2.1.1 Lotka-Volterra predator-prey Model (Host-Pathogen Model)

Let $H(t)$ be the concentration of the prey (host) population at time t and $P(t)$ be the concentration of predator (parasite) population, then the governing equation is

$$\left.\begin{array}{lll} \text{(Host/Prey)} & \frac{1}{H}\frac{dH}{dt} & = a_1 - b_1 P \\ \text{(Predator/Parasite)} & \frac{1}{P}\frac{dP}{dt} & = -a_2 + b_2 H \end{array}\right\} \tag{2.1.1}$$

where a_1, a_2, b_1, b_2 are all positive.
$a_1 \to$ growth rate pf prey in absence of predator,
$b_1 \to$ predation rate,
$a_2 \to$ death rate of predator in absence of prey,
$b_2 \to$ conversion rate

$$\begin{aligned} & \frac{P}{H}\frac{dH}{dP} = \frac{a_1 - b_1 P}{-a_2 + b_2 H} \\ \implies \quad & \frac{-a_2 + b_2 H}{H} dH = \frac{a_1 - b_1 P}{P} dP \\ \implies \quad & -a_2 \log H + b_2 H = a_1 \log P - b_1 P + c \\ \implies \quad & b_1 P + b_2 H - a_2 \log H - a_1 \log P = c \end{aligned}$$

Initially, $t = 0, H = H_0, P = P_0$. Therefore

$$\begin{aligned} & c = b_1 P_0 + b_2 H_0 - a_2 \log H_0 - a_1 \log P_0 \\ \implies \quad & b_1(P - P_0) + b_2(H - H_0) - a_2 \log \frac{H}{H_0} - a_1 \log \frac{P}{P_0} = 0 \\ \implies \quad & \log \left(\frac{H}{H_0}\right)^{a_2} + \log \left(\frac{P}{P_0}\right)^{a_1} + b_1(P - P_0) + b_2(H - H_0) = 0 \\ \implies \quad & \left(\frac{H}{H_0}\right)^{a_2} \left(\frac{P}{P_0}\right)^{a_1} = \text{Exp}[b_1(P - P_0) + b_2(H - H_0)] \end{aligned}$$

For equilibrium, we have $\frac{dH}{dt} = 0$, $\frac{dP}{dt} = 0$. So,

$$\begin{aligned} & H(a_1 - b_1 P) = 0 \quad \text{and} \quad P(-a_2 + b_2 H) = 0 \\ \implies & H = 0,\ P^* = \frac{a_1}{b_1} \quad \text{and} \quad P = 0,\ H^* = \frac{a_2}{b_2} \end{aligned}$$

Stability

$$V(H, P) = \begin{bmatrix} a_1 - b_1 P & -b_1 H \\ b_2 P & -a_2 + b_2 H \end{bmatrix}, \quad V(H^*, P^*) = \begin{bmatrix} 0 & -b_1 H^* \\ b_2 P^* & 0 \end{bmatrix}$$

The characteristic equation of $V(H^*, P^*)$ is (for interior equilibrium)

$$\begin{aligned} & \begin{vmatrix} 0 - \lambda & -b_1 H^* \\ b_2 P^* & 0 - \lambda \end{vmatrix} = 0 \\ \implies & \lambda^2 + b_1 b_2 H^* P^* = 0 \\ \implies & \lambda^2 + b_1 b_2 \cdot \frac{a_2}{b_2} \cdot \frac{a_1}{b_1} = 0 \\ \implies & \lambda^2 + a_1 a_2 = 0 \\ \implies & \lambda = \pm i\sqrt{a_1 a_2} \end{aligned}$$

Stability by Perturbation Method

Let $H = H^* + h$ and $P = P^* + p$ where $h, p > 0$ and are so small such that their powers and products can be neglected. Then

$$\begin{aligned} \frac{dh}{dt} &= a_1(H^* + h) - b_1(H^* + h)(P^* + p) \\ &= a_1 H^* + a_1 h - b_1 H^* P^* - b_1 h P^* - b_1 H^* p - b_1 h p \\ &= a_1 h - b_1 h P^* - b_1 H^* p \\ &= a_1 h - b_1 h \cdot \frac{a_1}{b_1} - b_1 p \cdot \frac{a_2}{b_2} \\ &= -\frac{a_2 b_1 p}{b_2} \end{aligned}$$

and

$$\begin{aligned} \frac{dp}{dt} &= -a_2(P^* + p) + b_2(H^* + h)(P^* + p) \\ &= -a_2 P^* - a_2 p + b_2 H^* P^* + b_2 H^* p + b_2 P^* h + b_2 h p \\ &= -a_2 p + b_2 H^* p + b_2 P^* h \\ &= -a_2 p + b_2 p \cdot \frac{a_2}{b_2} + b_2 h \cdot \frac{a_1}{b_1} \\ &= \frac{a_1 b_2 h}{b_1} \end{aligned}$$

Then

$$\frac{dh}{dp} = \frac{-\frac{a_2 b_1 p}{b_2}}{\frac{a_1 b_2 h}{b_1}} = -\frac{a_2 b_1^2}{a_1 b_2^2} \cdot \frac{p}{h}$$

which gives $a_1 b_2^2 h dh + a_2 b_1^2 p dp = 0$. Integrating we get

$$
\begin{aligned}
& a_1 b_2^2 \frac{h^2}{2} + a_2 b_1^2 \frac{p^2}{2} = A \\
\implies \quad & \frac{h^2}{\frac{2A}{a_1 b_2^2}} + \frac{p^2}{\frac{2A}{a_2 b_1^2}} = 1
\end{aligned}
$$

This is an equation of ellipse and so the phase plane is elliptic. Therefore, the system is stable at (H^*, P^*).

2.1.2 Gauss Competition Model

$$
\frac{1}{N_1}\frac{dN_1}{dt} = r_1 - a_{11}N_1 - a_{12}N_2 \tag{2.1.2}
$$

$$
\frac{1}{N_2}\frac{dN_2}{dt} = r_2 - a_{21}N_1 - a_{22}N_2 \tag{2.1.3}
$$

$N_1(t), N_2(t) \rightarrow$ Two competing species,
$r_1 \rightarrow$ growth rate of N_1,
$r_2 \rightarrow$ growth rate of N_2,
$a_{11}, a_{22} \rightarrow$ Intra species competition coefficient,
$a_{12}, a_{21} \rightarrow$ Inter species competition coefficient,
$a_{12} \rightarrow$ Effect of competition of N_2 on N_1,
$a_{21} \rightarrow$ Effect of competition of N_1 on N_2.

Equilibria

$$
\begin{aligned}
& \frac{dN_1}{dt} = 0, \quad \frac{dN_2}{dt} = 0 \\
\implies \quad & r_1 - a_{11}N_1 - a_{12}N_2 = 0, \;\; r_2 - a_{21}N_1 - a_{22}N_2 = 0 \\
\implies \quad & a_{11}N_1 + a_{12}N_2 = r_1, \;\; a_{21}N_1 + a_{22}N_2 = r_2
\end{aligned}
$$

Solving these two equations, we get

$$
\frac{N_1}{a_{22}r_1 - a_{12}r_2} = \frac{N_2}{a_{11}r_2 - a_{21}r_1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}
$$

Therefore, we get the solution

$$
N_1^* = \left(\frac{a_{22}r_1 - a_{12}r_2}{a_{11}a_{22} - a_{12}a_{21}}\right), \quad N_2^* = \left(\frac{a_{11}r_2 - a_{21}r_1}{a_{11}a_{22} - a_{12}a_{21}}\right)
$$

Now $a_{11}N_1 + a_{12}N_2 = r_1$ and $a_{21}N_1 + a_{22}N_2 = r_2$ can be expressed as

$$
\begin{aligned}
\frac{N_1}{\frac{r_1}{a_{11}}} + \frac{N_2}{\frac{r_1}{a_{12}}} = 1 \\
\frac{N_1}{\frac{r_2}{a_{21}}} + \frac{N_2}{\frac{r_1}{a_{22}}} = 1
\end{aligned}
$$

Stability

Let $N_1 = N_1^* + n_1, N_2 = N_2^*$ where n_1 is so small such that the square and higher powers can be neglected. Then from (2.1.2) we get

$$\begin{aligned}\frac{dn_1}{dt} &= r_1(N_1^* + n_1) - a_{11}(N_1^* + n_1)^2 - a_{12}N_2^*(N_1^* + n_1)\\ &= (r_1N_1^* - a_{11}{N_1^*}^2 - a_{12}N_1^*N_2^*) + r_1n_1 - 2a_{11}N_1^*n_1 - a_{11}n_1^2 - a_{12}n_1N_2^*\\ &= r_1n_1 - 2a_{11}N_1^*n_1 - a_{12}n_1N_2^*\\ &= n_1(r_1 - a_{11}N_1^* - a_{12}N_2^*) - a_{11}n_1N_1^*\\ &= -a_{11}n_1N_1^*\end{aligned}$$

$$\implies \quad \frac{dn_1}{n_1} = -a_{11}N_1^*dt$$

Integrating we get

$$\log n_1 = -a_{11}N_1^*t + \log A \quad \implies \quad n_1 = Ae^{-a_{11}N_1^*t}$$

We can observe that $n_1 \to 0$ as $t \to \infty$. Therefore, (N_1^*, N_2^*) is stable.

Stability of equilibrium solution of Gauss's equation

Recall (2.1.2) and (2.1.3)

$$\begin{aligned}\frac{1}{N_1}\frac{dN_1}{dt} &= r_1 - a_{11}N_1 - a_{12}N_2\\ \frac{1}{N_2}\frac{dN_2}{dt} &= r_2 - a_{21}N_1 - a_{22}N_2\end{aligned}$$

and (N_1^*, N_2^*) being the solution of $\frac{dN_1}{dt} = 0$, $\frac{dN_2}{dt} = 0$ satisfy

$$\begin{aligned}r_1 - a_{11}N_1^* - a_{12}N_2^* &= 0,\\ r_2 - a_{21}N_1^* - a_{22}N_2^* &= 0\end{aligned}$$

Now let $N_1 = N_1^* + n_1$ and $N_2 = N_2^* + n_2$ where $n_1, n_2 > 0$ and are so small such that their powers and products can be neglected. Then from (2.1.2) we get

$$\begin{aligned}\frac{dn_1}{dt} &= r_1(N_1^* + n_1) - a_{11}(N_1^* + n_1)^2 - a_{12}(N_1^* + n_1)(N_2^* + n_2)\\ &= (r_1N_1^* - a_{11}{N_1^*}^2 - a_{12}N_1^*N_2^*) + r_1n_1 - 2a_{11}N_1^*n_1 - a_{11}n_1^2 - a_{12}n_1N_2^* - a_{12}n_2N_1^* - a_{12}n_1n_2\\ &= r_1n_1 - 2a_{11}N_1^*n_1 - a_{12}n_1N_2^* - a_{12}n_2N_1^*\\ &= n_1(r_1 - a_{11}N_1^* - a_{12}N_2^*) - a_{11}n_1N_1^* - a_{12}n_2N_1^*\\ &= -(a_{11}n_1 + a_{12}n_2)N_1^*\end{aligned}$$

and from (2.1.3) we obtain

$$\begin{aligned}\frac{dn_2}{dt} &= r_2(N_2^* + n_2) - a_{21}(N_1^* + n_1)(N_2^* + n_2) - a_{22}(N_2^* + n_2)^2\\ &= (r_2N_2^* - a_{21}N_1^*N_2^* - a_{22}{N_2^*}^2) + r_2n_2 - 2a_{22}N_2^*n_2 - a_{21}n_1N_2^* - a_{21}n_2N_1^* - a_{22}n_2^2\\ &= r_2n_2 - 2a_{22}N_2^*n_2 - a_{21}n_1N_2^* - a_{21}n_2N_1^*\\ &= n_2(r_2 - a_{21}N_1^* - a_{22}N_2^*) - a_{22}n_2N_2^* - a_{21}n_1N_2^*\\ &= -(a_{22}n_2 - a_{21}n_1)N_2^*\end{aligned}$$

We can represent these values of $\frac{dn_1}{dt}$ and $\frac{dn_2}{dt}$ as

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -a_{11}N_1^* & -a_{12}N_1^* \\ -a_{21}N_2^* & -a_{22}N_2^* \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Thus we have got a matrix equation

$$\frac{d}{dt}N = AN \qquad \text{where} \quad A = \begin{pmatrix} -a_{11}N_1^* & -a_{12}N_1^* \\ -a_{21}N_2^* & -a_{22}N_2^* \end{pmatrix}, N = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Now the eigen value equation of the matrix A is $Av_l = \lambda_l v_l$ where λ_l's and v_l's (for $l = 1, 2$) are the eigen values and eigen vectors of A respectively.

Let $N = \sum_l v_l e^{\lambda_l t}$, then

$$\frac{dN}{dt} = \sum_l v_l \lambda_l e^{\lambda_l t} = \sum_l A v_l e^{\lambda_l t} = A \sum_l v_l e^{\lambda_l t} = AN$$

Therefore, $N = \sum_l v_l e^{\lambda_l t}$ is a solution of the matrix equation. Now

$$N = \sum_l v_l e^{\lambda_l t} \quad \Longrightarrow \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \sum_l \begin{pmatrix} v_{l_1} \\ v_{l_2} \end{pmatrix} e^{\lambda_l t}$$

$$n_1 = \sum_l v_{l_1} e^{\lambda_l t} \;\; \text{and} \;\; n_2 = \sum_l v_{l_2} e^{\lambda_l t}$$

Let $\lambda_l = (\text{Re } \lambda_l) + i(\text{Im } \lambda_l)$, then

$$n_1 = \sum_l v_{l_1} e^{(\text{Re } \lambda_l)t} \left[\cos\ (\text{Im } \lambda_l)t + i \sin\ (\text{Im } \lambda_l)t\right]$$

So, if Re $(\lambda_l) < 0$ for $l = 1, 2$, then $n_1, n_2 \to 0$ as $t \to \infty$. In this case we have stable equilibrium.

2.2 Several species populations

Let $N_i(t)$, for $i = 1, 2, \ldots, n$, be the number of individuals of the i-th species, then the general Lotka-Volterra type model for n-species population is given by

$$\begin{aligned} \frac{dN_i}{dt} &= k_i N_i - \sum_{i,j=1}^{n} \alpha_{ij} N_i N_j \\ &= k_i N_i - \alpha_{ii} N_i^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \alpha_{ij} N_i N_j \\ &= k_i N_i \left(1 - \frac{N_i}{\frac{k_i}{\alpha_{ii}}}\right) - \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \alpha_{ij} N_i N_j \end{aligned}$$

where K_i is the growth rate of the i-th species, α_{ii} is a self-interaction parameter for the i-th species, α_{ij} is the interaction parameter denoting the effect of j-th species on the i-th species and $\frac{k_i}{\alpha_{ii}}$ is the carrying capacity.

Equilibrium is given by

$$\begin{aligned} &\frac{dN_i}{dt} = 0 \\ \implies \quad &k_i N_i - \sum_{i,j=1}^{n} \alpha_{ij} N_i N_j = 0 \\ \implies \quad &k_i = \sum_{j=1}^{n} \alpha_{ij} N_j^* = 0 \end{aligned}$$

This can be written as

$$AN = K \quad \implies \quad N = A^{-1}K \quad (\text{provided } \; |A| \neq 0)$$

where

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}, \quad K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} N_1^* \\ N_2^* \\ \vdots \\ N_n^* \end{pmatrix}.$$

Stability Analysis

Let $N_i = N_i^* + x_i$, for $i = 1, 2, \ldots, n$, where x_i's are so small such that their powers and products can be neglected.

$$\begin{aligned} &\frac{dN_i}{dt} = k_i N_i - \sum_{i,j=1}^{n} \alpha_{ij} N_i N_j \\ \implies \quad &\frac{d}{dt}(N_i^* + x_i) = k_i(N_i^* + x_i) - \sum_{i,j=1}^{n} \alpha_{ij}(N_i^* + x_i)(N_j^* + x_j) \\ \implies \quad &\frac{dx_i}{dt} = (N_i^* + x_i)[k_i - \sum_{j=1}^{n} \alpha_{ij}(N_j^* + x_j)] \end{aligned}$$

Since, at equilibrium,

$$k_i = \sum_{j=1}^{n} \alpha_{ij} N_j^*$$

we have

$$\begin{aligned} \frac{dx_i}{dt} &= (N_i^* + x_i)(-\sum_{j=1}^{n} \alpha_{ij} x_j) \\ &= -N_i^* \sum_{j=1}^{n} \alpha_{ij} x_j = \sum_{j=1}^{n} \beta_{ij} x_j \end{aligned}$$

where $\beta_{ij} = -N_i^* \alpha_{ij}$. In matrix notation

$$\frac{dX}{dt} = BX$$

where

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix}.$$

Using the eigen equation $Bv_l = \lambda_l v_l$, we get that a solution of the matrix equation is given by

$$X = \sum_l v_l e^{\lambda_l t} \quad \Longrightarrow \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_l \begin{pmatrix} v_{l_1} \\ \vdots \\ v_{l_n} \end{pmatrix} e^{\lambda_l t}$$

Therefore, $x_i = \sum_l v_{l_i} e^{\lambda_l t}$. Putting $\lambda_l = (\text{Re } \lambda_l) + i(\text{Im } \lambda_l)$, we get

$$x_i = \sum_l v_{l_i} e^{(\text{Re } \lambda_l)t} \left[\cos\ (\text{Im } \lambda_l)t + i \sin\ (\text{Im } \lambda_l)t\right]$$

- If Re$(\lambda_l) < 0$ for all l, then $x_i \to 0$ as $t \to \infty$ and stability arise.
- If Re$(\lambda_l) > 0$ for all l, then $x_i \to \infty$ as $t \to \infty$ and the system is unstable.
- If Re$(\lambda_l) < 0$ for all l except $l = m$, then $x_i \to v_{m_i} e^{(\text{Re } \lambda_m)t} \left[\cos\ (\text{Im } \lambda_m)t + i \sin\ (\text{Im } \lambda_m)t\right]$ as $t \to \infty$ and the system is unstable.

For $i \neq j$,

(i) $\alpha_{ij} > 0 \quad \Rightarrow \quad$ Competition

(ii) $\alpha_{ij} < 0 \quad \Rightarrow \quad$ Symbiosis

(iii) $\alpha_{ij} = -\alpha_{ji} \quad \Rightarrow \quad$ Prey-predator

2.2.1 Stability of Gompertz growth model for *n*-species

Let us consider the model equation described by

$$\frac{1}{N_i} \frac{dN_i}{dt} = k_i - \sum_{j=1}^{n} \alpha_{ij} \ln N_j$$

This model can be solved by putting $\ln N_i = x_i$. Then

$$\frac{dx_i}{dt} = \frac{1}{N_i} \frac{dN_i}{dt} = k_i - \sum_{j=1}^{n} \alpha_{ij} x_j$$

Units 3 & 4

3.1 Leslie-Gower Predator-Prey Model

The dynamic relationship between predators and preys has long been and will continue to be one of the dominant themes due to its universal existence and importance. Leslie introduced the following two species Leslie-Gower predator-prey model:

$$\begin{cases} \dot{x}(t) = (r_1 - b_1 x)x - p(x)y \\ \dot{y}(t) = \left(r_2 - \frac{a_2 y}{x}\right) y \end{cases} \tag{3.1.1}$$

where $x(t), y(t)$ are population density of the prey and predator at time t respectively, r_1, r_2 are the intrinsic growth rates of prey and predator respectively, b_1 measures the strength of competition among individuals of species x, $\frac{r_1}{b_1}$ is the carrying capacity of the prey in the absence of the predator. The predator consumes the prey according to the functional response $p(x)$ and grows logistically with growth rate r_2 and carrying capacity $\frac{r_2 x}{a_2}$ proportional to the population size of the prey.

a_2 is measure of the food quantity that the prey provides and converted to predator birth. The term $\frac{y}{x}$ is the Leslie-Gower term which measures the loss in the predator population due to rarity of its favourite food.

3.2 Modified Leslie-Gower and Holling type-II schemes

$$\begin{cases} \dot{x}(t) = \left(r_1 - b_1 x - \frac{a_1 y}{x+k_1}\right) x, \\ \dot{y}(t) = \left(r_2 - \frac{a_2 y}{x+k_2}\right) y \end{cases} \tag{3.2.1}$$

with $x(0) \geq 0$ and $y(0) \geq 0$, where r_1, b_1, r_2, a_2 have the same meaning as in the system (3.1.1), a_1 is the maximum value which per capita reduction rate of x can attain and k_1, k_2 measure the extent to which environment provides protection to prey x and predator y respectively.

3.2.1 Global stability

We shall prove the global stability of system (3.2.1) by constructing a suitable Lyapunov function. First of all, it is verify that the system (3.2.1) has three trivial equilibria $E_0 = (0, 0)$, $E_1 = \left(\frac{r_1}{b_1}, 0\right)$ and $E_2 = \left(0, \frac{r_2 k_2}{a_2}\right)$.

Theorem 3.2.1. Let us assume the following condition:

$$\frac{r_2 k_2}{a_2} < \frac{r_1 k_1}{a_1}$$

Then the system (3.2.1) has a unique interior equilibrium $E^*(x^*, y^*)$ (that is, $x^* > 0$ and $y^* > 0$).

Proof. From system (3.2.1), such a point satisfies

$$(r_1 - b_1 x^*)(x^* + k_1) = a_1 y^* \tag{3.2.2}$$

$$y^* = \frac{r_2(x^* + k_2)}{a_2} \tag{3.2.3}$$

Now

$$\begin{aligned} & r_1 x^* - b_1 {x^*}^2 + r_1 k_1 - b_1 k_1 x^* = \frac{a_1 r_2 (x^* + k_2)}{a_2} \\ \implies & -a_2 b_1 {x^*}^2 + (r_1 a_2 - b_1 k_1 a_2 - a_1 r_2) x^* + a_2 r_1 k_1 - a_1 r_2 k_2 = 0 \\ \implies & a_2 b_1 {x^*}^2 + (a_1 r_2 - r_1 a_2 + b_1 k_1 a_2) x^* + a_1 r_2 k_2 - a_2 r_1 k_1 = 0 \end{aligned}$$

Solving the above equation we get

$$x^*_{\pm} = \frac{1}{2a_2 b_1}(-(a_1 r_2 - r_1 a_2 + b_1 k_1 a_2) \pm \Delta^{\frac{1}{2}})$$

where

$$\Delta = (a_1 r_2 - r_1 a_2 + b_1 k_1 a_2)^2 - 4a_2 b_1 (a_1 r_2 k_2 - a_2 r_1 k_1)$$

Clearly Δ is non-negative if $\frac{r_2 k_2}{a_2} < \frac{r_1 k_1}{a_1}$ holds. Moreover, simple algebraic calculations show that under our assumed condition $x^*_+ > 0$ and $x^*_- < 0$. Therefore, the system (3.2.1) possesses a unique interior equilibrium $E^*(x^*, y^*)$ given by

$$\begin{aligned} x^* &= \frac{1}{2a_2 b_1}(-(a_1 r_2 - r_1 a_2 + b_1 k_1 a_2) + \Delta^{\frac{1}{2}}) \\ y^* &= \frac{r_2(x^* + k_2)}{a_2} \end{aligned}$$

□

Linear analysis of model (3.2.1) shows that if $r_1 \leq r_2$ and $k_1 \geq k_2$, then $E^*(x^*, y^*)$ is locally stable.

Theorem 3.2.2. The interior equilibrium $E^*(x^*, y^*)$ is globally asymptotically stable if

$$L_1 < \frac{r_1 k_1}{2a_1} \tag{3.2.4}$$

$$k_1 < 2k_2 \tag{3.2.5}$$

$$4(r_1 + b_1 k_+ 1) < a_1 \tag{3.2.6}$$

where

$$L_1 = \frac{1}{4a_2 b_1}(a_2 r_1 (r_1 + 4) + (r_2 + 1)^2 (r_1 + b_1 k_2)).$$

Proof. The proof is based on a positive definite Lyapunov function. Let $V(x, y) = V_1(x, y) + V_2(x, y)$ where

$$\begin{aligned} & V_1(x, y) = (x^* + k_1)\left(x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right) \\ \text{and} \quad & V_2(x, y) = \frac{a_1(x^* + k_2)}{a_2}\left(y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right) \end{aligned}$$

This function is defined and continuous on $\text{Int}(\mathbb{R}^2_+)$. It can be easily verified that the function $V(x, y)$ is zero at equilibrium (x^*, y^*) and is positive for all other positive values of x and y, and thus $E^*(x^*, y^*)$ is the global minimum of V.

The time derivative of V_1 along the solution of system (3.2.1) is

$$\frac{dV_1}{dt} = \frac{(x^* + k_1)(x - x^*)}{x}\left(r_1 - b_1 x - \frac{a_1 y}{x + k_1}\right) x$$

and using (3.2.2), we get

$$\begin{aligned}
\frac{dV_1}{dt} &= (x^* + k_1)(x - x^*)\left(-b_1(x - x^*) + \frac{a_1 y^*}{x^* + k_1} - \frac{a_1 y}{x + k_1}\right) \\
&= (x^* + k_1)(x - x^*)\left(-b_1(x - x^*) + \frac{a_1 y^*(x + k_1) - a_1 y(x^* + k_1)}{(x + k_1)(x^* + k_1)}\right) \\
&= (x^* + k_1)(x - x^*)\left(-b_1(x - x^*) + \frac{-a_1 k_1(y - y^*) - a_1 x(y - y^*) + a_1 y(x - x^*)}{(x + k_1)(x^* + k_1)}\right)
\end{aligned}$$

Similarly,

$$\frac{dV_2}{dt} = \frac{a_1(x^* + k_2)(y - y^*)}{a_2 y}\left(r_2 - \frac{a_2 y}{x + k_2}\right) y$$

and we use (3.2.3) to get

$$\begin{aligned}
\frac{dV_2}{dt} &= \frac{a_1(x^* + k_2)(y - y^*)}{a_2}\left(\frac{a_2 y^*}{x^* + k_2} - \frac{a_2 y}{x + k_2}\right) \\
&= a_1(x^* + k_2)(y - y^*)\left(\frac{a_2 y^*(x + k_2) - y(x^* + k_2)}{(x + k_2)(x^* + k_2)}\right) \\
&= a_1(x^* + k_2)(y - y^*)\left(\frac{-k_2(y - y^*) - x(y - y^*) + y(x - x^*)}{(x + k_2)(x^* + k_2)}\right)
\end{aligned}$$

Now, computing $\frac{dV}{dt}$ with the help of $\frac{dV_1}{dt}$ and $\frac{dV_2}{dt}$ yields

$$\frac{dV}{dt} = \left(-b_1(x^* + k_1) + \frac{a_1 y}{x + k_1}\right)(x - x^*)^2 + \left(-a_1 + \frac{a_1 y}{x + k_2}\right)(x - x^*)(y - y^*) - a_1(y - y^*)^2 \quad (3.2.7)$$

The above equation can be written as

$$\frac{dV}{dt} = -X^t M X$$

where

$$M = \begin{pmatrix} -g(x, y) & -h(x, y) \\ -h(x, y) & a_1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix},$$

$$\begin{aligned}
g(x, y) &= -b_1(x^* + k_1) + \frac{a_1 y}{x + k_1} \\
h(x, y) &= \frac{1}{2}\left(-a_1 + \frac{a_1 y}{x + k_2}\right)
\end{aligned}$$

From (3.2.7), it is obvious that $\frac{dV}{dt} < 0$ if the matrix M is positive definite. Now from Sylvester's criteria we know that a matrix is positive definite if and only if all of its upper-left submatrices are of positive determinants. Here, since $a_1 > 0$, M is positive definite if and only if

(i) $g(x,y) < 0$ and

(ii) $\phi(x,y) = -a_1 g(x,y) - h^2(x,y) < 0$.

Proof of (i):

$$
\begin{aligned}
g(x,y) &= -b_1(x^* + k_1) + \frac{a_1 y}{x + k_1} \\
&= -r_1 + \frac{a_1 y^*}{x^* + k_1} + \frac{a_1 y}{x + k_1} \quad \text{[using (3.2.2)]}
\end{aligned}
$$

Let $\mathcal{A}$ be the set defined by

$$
\mathcal{A} = \left\{ (x,y) \in \mathbb{R}_+^2 : \; 0 \le x \le \frac{r_1}{b_1}, \; 0 \le x + y \le L_1 \right\}
$$

where L_1 is as previously defined. Then $\mathcal{A}$ is positively invariant and all solutions of (3.2.1) initiating in $\mathbb{R}_+^2$ are ultimately bounded with respect to $\mathbb{R}_+^2$ and eventually enter the attracting set $\mathcal{A}$.

So, as $\mathcal{A}$ is an attracting positively invariant set and in $\mathcal{A}$, all solutions satisfy $0 \le x \le \frac{r_1}{b_1}$ and $0 \le x + y \le L_1$, then

$$
g(x,y) \le -r_1 + \frac{a_1}{k_1}(y + y^*) \le -r_1 + \frac{2a_1 L_1}{k_1}
$$

Therefore, if (3.2.4) holds, then for all $(x,y) \in \mathcal{A}$, $g(x,y) < 0$ for all $t \ge 0$.

Proof of (ii):

$$
\phi(x,y) = -a_1 \left(-b_1(x^* + k_1) + \frac{a_1 y}{x + k_1} \right) - \frac{1}{4} \left(-a_1 + \frac{a_1 y}{x + k_2} \right)^2
$$

Then

$$
\begin{aligned}
\frac{\partial \phi(x,y)}{\partial y} &= \frac{-a_1^2}{x + k_1} - \frac{1}{2} \left(\frac{-a_1^2}{x + k_2} + \frac{a_1^2 y}{(x + k_2)^2} \right), \\
\frac{\partial^2 \phi(x,y)}{\partial y^2} &= -\frac{a_1^2}{2(x + k_2)^2} < 0
\end{aligned}
$$

Hence $\frac{\partial \phi(x,y)}{\partial y}$ is strictly decreasing in $\mathbb{R}_+$ with respect to y.

Now,

$$
\left. \frac{\partial \phi(x,y)}{\partial y} \right|_{y=0} = \frac{-a_1^2}{x + k_1} + \frac{a_1^2}{2(x + k_2)} = \frac{a_1^2(-x - 2k_2 + k_1)}{2(x + k_1)(x + k_2)}
$$

Consequently, if (3.2.5) holds,

$$
\left. \frac{\partial \phi(x,y)}{\partial y} \right|_{y=0} < 0
$$

in $\mathbb{R}_+$ and so $\phi(x,y)$ is strictly decreasing in $\mathbb{R}_+$. This yields, for $(x,y) \in \mathcal{A}$,

$$
\phi(x,y) < a_1 b_1 (x^* + k_1) - \frac{1}{4} a_1^2
$$

As $0 \le x^* \le \frac{r_1}{b_1}$, then $\phi(x,y) < a_1(r_1 + b_1 k_1 - (1/4)a_1)$, and finally due to (3.2.6), for all $(x,y) \in \mathcal{A}$, we get $\phi(x,y) < 0$.

It follows that if the hypothesis of (3.2.4), (3.2.5) and (3.2.6) are satisfied, then $\frac{dV}{dt} < 0$ along all trajectories in the first quadrant except (x^*, y^*), so that $E^*(x^*, y^*)$ is globally asymptotically stable. □

Units 5 & 6

5.1 Epidemic Models

A basic epidemic model can be used to understand the dynamics of epidemic. We will separate the population into three classes determined by the state of individual relative to the diseases.

Those who do not have the disease and can *potentially get* the disease are called **susceptible**.

Individuals who can *infect others* are called **infective**.

Individuals who have either died or recovered and no longer can infect the others are called **recovered**.

Possible transmission is as follows:

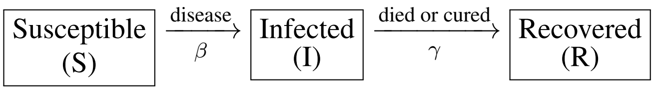


Fig: Compartmental model

$S(t), I(t)$ and $R(t)$ represent the number of individuals in the susceptible, infected and recovered class respectively.

We assume that there is no immigration or size emigration to the total population (N), that is,

$$S + I + R = \text{constant} = N$$

Let β be the transmission rate. We assume that the rate of infection is βSI, that is, the product of susceptible and infected individuals with the transmission rate. The removal rate is assumed to be constant. Based on these we can write the model as:

$$\text{Kermack-McKendrick model} \begin{cases} \frac{dS}{dt} = -\beta SI \\ \frac{dI}{dt} = \beta SI - \gamma I \\ \frac{dR}{dt} = \gamma I \end{cases} \qquad (\ S\text{-}I\text{-}R \text{ model})$$

or,

$$\frac{dS}{dt} = -\beta SI + \gamma I$$
$$\frac{dI}{dt} = \beta SI - \gamma I \qquad (\ S\text{-}I\text{-}S \text{ model})$$

Epidemic will persists until:

$$\begin{aligned}
& \frac{dI}{dt} > 0 \\
\implies \quad & (\beta S - \gamma) I > 0 \\
\implies \quad & \beta S - \gamma > 0 \\
\implies \quad & \frac{\beta S}{\gamma} > 0
\end{aligned}$$

Initially $S = N$ and let

$$\boxed{R_0 = \frac{\beta N}{\gamma}}$$

R_0 is called the **basic reproduction number**. Then from the above equations we can deduce that
if $R_0 > 1$, the disease will persist,
if $R_0 < 1$, the disease will die out.

We see that there is maximum size of population in which there can be an epidemic, so for an epidemic we must have

$$\boxed{N > \frac{\gamma}{\beta}}$$

Now from the S-I-R model, we get

$$\begin{aligned}
& \frac{dI}{dS} = -1 + \frac{\gamma}{\beta S} \\
\implies \quad & dI = \left(-1 + \frac{\gamma}{\beta S}\right) \\
\implies \quad & I = \frac{\gamma}{\beta} \ln S - S + \ln C
\end{aligned}$$

Initially $I = 0, S = N$, so

$$0 = \frac{\gamma}{\beta} \ln N - N + \ln C$$

Therefore,

$$\begin{aligned}
I(t) &= \frac{\gamma}{\beta} \ln S - S + N - \frac{\gamma}{\beta} \ln N \\
&= \frac{\gamma}{\beta} \ln \frac{S}{N} + (N - S) \qquad (5.1.1)
\end{aligned}$$

When $I = 0$, we get that the *transcendental equation*

$$0 = \frac{\gamma}{\beta} \ln \frac{S}{N} + (N - S) \qquad (5.1.2)$$

We can not easily solve the equation (5.1.2). We see that the right hand side of (5.1.2) is zero when $S = N$ and approaches negative infinity as $S \to 0$. Thus the other value of S must be much greater than zero.

5.2 *S-E-I-R* Model

$$\frac{dS}{dt} = \mu N - \beta c \frac{SI}{N} - \mu S$$
$$\frac{dE}{dt} = \beta c \frac{SI}{N} - (\sigma + \mu)E$$
$$\frac{dI}{dt} = \sigma E - (\gamma + \mu)I$$
$$\frac{dR}{dt} = \gamma I - \mu R$$

$S + E + I + R =$ constant $= N \rightarrow$ total population size
$\mu \rightarrow$ birth and death rate
$\beta \rightarrow$ transmission rate
$c \rightarrow$ compartmental constant
$\sigma \rightarrow$ transmission rate from exposed to infected class
$\gamma \rightarrow$ recovery rate
Since, $S+E+I+R =$ constant, the last equation is redundant. Hence, we try to solve the first three equations and find (S^*, E^*, I^*).

For endemic equilibrium

$$\begin{aligned}
\frac{dI}{dt} = 0 \quad &\implies \quad \sigma E^* - (\gamma + \mu)I^* = 0 \\
&\implies \quad E^* = \frac{(\gamma + \mu)I^*}{\sigma} \\
\frac{dE}{dt} = 0 \quad &\implies \quad \beta c \frac{S^* I^*}{N} = (\sigma + \mu)E^* \\
&\implies \quad \beta c \frac{S^* I^*}{N} = \frac{(\sigma + \mu)(\gamma + \mu)}{\sigma} I^* \\
&\implies \quad S^* = \frac{N(\sigma + \mu)(\gamma + \mu)}{\beta c \sigma} \\
\frac{dS}{dt} = 0 \quad &\implies \quad \mu N = \beta c \frac{S^* I^*}{N} + \mu S^* \\
&\implies \quad \frac{\mu N}{S^*} = \frac{\beta c}{N} I^* + \mu \\
&\implies \quad I^* = \frac{N}{\beta c}\left(\frac{\mu N}{S^*} - \mu\right)
\end{aligned}$$

Now $I^* > 0$ implies

$$\begin{aligned}
&\frac{N\mu}{\beta c}\left(\frac{N}{S^*} - 1\right) > 0 \\
\implies \quad &\frac{N}{S^*} - 1 > 0 \\
\implies \quad &N > S^* = \frac{N(\sigma + \mu)(\gamma + \mu)}{\beta c \sigma} \\
\implies \quad &\frac{\beta c \sigma}{(\sigma + \mu)(\gamma + \mu)} > 1
\end{aligned}$$

Therefore,

$$R_0 = \frac{\beta c \sigma}{(\sigma+\mu)(\gamma+\mu)}$$

Stable if $R_0 < 1 \leftarrow (S, 0, 0) \rightarrow$ disease free,
Stable if $R + 0 > 1 \leftarrow (S^*, E^*, I^*) \rightarrow$ endemic equilibrium

For the *S-E-I-R* Model, we have

$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \beta c \frac{SI}{N} - \mu S = f_1(S, E, I) \\
\frac{dE}{dt} &= \beta c \frac{SI}{N} - (\sigma+\mu)E = f_2(S, E, I) \\
\frac{dI}{dt} &= \sigma E - (\gamma+\mu)I = f_3(S, E, I)
\end{aligned}$$

Stability

For stability, consider the variational matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial I} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial I} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial I} \end{bmatrix} = \begin{bmatrix} -\frac{\beta c I^*}{N} - \mu & 0 & -\frac{\beta c S^*}{N} \\ \frac{\beta c I^*}{N} & -(\sigma+\mu) & \frac{\beta c S^*}{N} \\ 0 & \sigma & -(\gamma+\mu) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\mu N}{S^*} & 0 & -\frac{\beta c S^*}{N} \\ \frac{\beta c I^*}{N} & -(\sigma+\mu) & \frac{\beta c S^*}{N} \\ 0 & \sigma & -(\gamma+\mu) \end{bmatrix}$$

$$\left[\text{Since, } I^* = \frac{N}{\beta c}\left(\frac{\mu N}{S^*} - \mu\right) \implies -\frac{\beta c I^*}{N} - \mu = -\frac{\mu N}{S^*}\right]$$

The characteristic equation is

$$\begin{vmatrix} -\frac{\mu N}{S^*} - \lambda & 0 & -\frac{\beta c S^*}{N} \\ \frac{\beta c I^*}{N} & -(\sigma+\mu) - \lambda & \frac{\beta c S^*}{N} \\ 0 & \sigma & -(\gamma+\mu) - \lambda \end{vmatrix} = 0$$

$$\implies \left(-\frac{\mu N}{S^*} - \lambda\right)\left[\{-(\sigma+\mu) - \lambda\}\{-(\gamma+\mu) - \lambda\} - \frac{\beta c S^* \sigma}{N}\right] - \frac{\beta c S^*}{N}\frac{\sigma \beta c I^*}{N} = 0$$

$$\implies -\left(\frac{\mu N}{S^*} + \lambda\right)\left[\{(\sigma+\mu) + \lambda\}\{(\gamma+\mu) + \lambda\} - \frac{\beta c S^* \sigma}{N}\right] - \frac{\sigma \beta^2 c^2 S^* I^*}{N^2} = 0$$

$$\implies \left(\frac{\mu N}{S^*} + \lambda\right)\left[(\sigma+\mu)(\gamma+\mu) + (\gamma+\sigma+2\mu)\lambda + \lambda^2 - \frac{\beta c S^* \sigma}{N}\right] + \frac{\sigma \beta^2 c^2 S^* I^*}{N^2} = 0$$

$$\implies \lambda^3 + \left[(\gamma+\sigma+2\mu) + \frac{\mu N}{S^*}\right]\lambda^2 + \left[(\sigma+\mu)(\gamma+\mu) + \frac{\mu N}{S^*}(\gamma+\sigma+2\mu) - \frac{\beta c S^* \sigma}{N}\right]\lambda$$
$$+ \left[\frac{\mu N}{S^*}(\sigma+\mu)(\gamma+\mu) - \beta c \sigma \mu + \frac{\sigma \beta^2 c^2 S^* I^*}{N^2}\right] = 0$$

By Routh Hurwitz criteria for stability, we get

$$(\gamma+\sigma+2\mu) + \frac{\mu N}{S^*} > 0$$

and

$$
\begin{aligned}
&\frac{\mu N}{S^*}(\sigma+\mu)(\gamma+\mu) - \beta c\sigma\mu + \frac{\sigma\beta^2 c^2 S^* I^*}{N^2} > 0 \\
\implies & \frac{\mu N}{\frac{N(\sigma+\mu)(\gamma+\mu)}{\beta c\sigma}}(\sigma+\mu)(\gamma+\mu) - \beta c\sigma\mu + \frac{\sigma\beta c}{N}\left(\frac{\beta c S^* I^*}{N^2}\right) > 0 \\
\implies & \beta c\sigma\mu - \beta c\sigma\mu + \frac{\sigma\beta c}{N}(\mu N - \mu S^*) > 0 \\
\implies & \beta c\sigma\mu - \frac{\sigma\beta c\mu S^*}{N} > 0 \\
\implies & \beta c\sigma\mu - \frac{\sigma\beta c\mu}{N}\frac{N(\sigma+\mu)(\gamma+\mu)}{\beta c\sigma} > 0 \\
\implies & \beta c\sigma\mu - \mu(\sigma+\mu)(\gamma+\mu) > 0 \\
\implies & \frac{\beta c\sigma}{(\sigma+\mu)(\gamma+\mu)} > 1
\end{aligned}
$$

Therefore, $R_0 > 1$ is the criteria for stability.

5.3 Eco-epidemic Model

Let us consider a prey-predator model with infection prey population.

Assumptions:

1. The total population density of prey is N.
2. The population density of predator is F.
3. In the absence of disease, the prey population follows logistic growth, that is,
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{5.3.1}$$
4. In the presence of disease, the prey population is divided into two groups: susceptible prey (R) and infected prey (U).
5. Only the susceptible preys are capable of reproducing with logistic law
$$
\begin{aligned}
\frac{dR}{dt} &= aR\left(1 - \frac{R}{K}\right) \\
&= aR - bR^2 \quad \text{where } b = \frac{a}{K}
\end{aligned} \tag{5.3.2}
$$
6. The disease is only spreading among preys.
7. A susceptible prey becomes infected under the attacks of many viruses. The attacking rate as well as predation rate follows the *law of mass-action*.

Considering the above assumptions we can write the system as:

$$
\begin{aligned}
\text{Susceptible prey} \quad \frac{dR}{dt} &= aR - bR^2 - cFR - \lambda UR \; = f_1(R, U, F) \\
\text{Infected prey} \quad \frac{dU}{dt} &= \lambda UR - dUF - eU \; = f_2(R, U, F) \\
\text{Predator} \quad \frac{dF}{dt} &= cFR + dUF - fF \; = f_3(R, U, F)
\end{aligned}
$$

$c \rightarrow$ predation rate,
$\lambda \rightarrow$ rate of infection,
$e \rightarrow$ death rate of infected prey,
$f \rightarrow$ death rate of predator.

The types of equilibrium points are

- $(R, 0, 0)$
- $(R, U, 0) \rightarrow$ predator-free equilibrium,
- $(R, 0, F) \rightarrow$ disease-free prey predator equilibrium,
- $(R, U, F) \rightarrow$ interior equilibrium

For equilibrium,

$$\frac{dR}{dt} = 0 \implies aR - bR^2 - cFR - \lambda UR = 0$$
$$\implies a - bR - cF - \lambda U = 0$$
$$\implies \boxed{bR + \lambda U + cF = a} \tag{5.3.3}$$

$$\frac{dU}{dt} = 0 \implies \lambda UR - dUF - eU = 0$$
$$\implies \lambda R - dF - e = 0$$
$$\implies \boxed{\lambda R + 0U - dF = e} \tag{5.3.4}$$

$$\frac{dF}{dt} = 0 \implies cFR + dUF - fF = 0$$
$$\implies \boxed{cR + dU + 0F = f} \tag{5.3.5}$$

Now from (5.3.3), (5.3.4) and (5.3.5) applying Cramer's rule, we get

$$A = \begin{pmatrix} b & \lambda & c \\ \lambda & 0 & -d \\ c & d & 0 \end{pmatrix}; \quad \det A = bd^2 - \lambda cd + c\lambda d = bd^2,$$

$$A_1 = \begin{pmatrix} a & \lambda & c \\ e & 0 & -d \\ f & d & 0 \end{pmatrix}; \quad \det A_1 = ad^2 - \lambda fd + ced = d(ad + ce - f\lambda)$$

$$\therefore \boxed{R^* = \frac{(ad + ce - f\lambda)}{bd}}$$

$$A_2 = \begin{pmatrix} b & a & c \\ \lambda & e & -d \\ c & f & 0 \end{pmatrix}; \quad \det A_2 = bdf - acd + c\lambda f - ce^2 = f(bd + \lambda c) - c(ad + ce)$$

$$\therefore \boxed{U^* = \frac{f(bd + \lambda c) - c(ad + ce)}{bd^2}}$$

$$A_3 = \begin{pmatrix} b & \lambda & a \\ \lambda & 0 & e \\ c & d & f \end{pmatrix}; \quad \det A_3 = -bde - \lambda^2 f + \lambda ce + a\lambda d = \lambda(ad - \lambda f) + e(\lambda c - bd)$$

$$\therefore \boxed{F^* = \frac{\lambda(ad - \lambda f) + e(\lambda c - bd)}{bd^2}}$$

(R^*, U^*, F^*) as determined above gives the interior equilibrium.

Stability

The variational matrix is

$$\begin{bmatrix} \frac{\partial f_1}{\partial R} & \frac{\partial f_1}{\partial U} & \frac{\partial f_1}{\partial F} \\ \frac{\partial f_2}{\partial R} & \frac{\partial f_2}{\partial U} & \frac{\partial f_2}{\partial F} \\ \frac{\partial f_3}{\partial R} & \frac{\partial f_3}{\partial U} & \frac{\partial f_3}{\partial F} \end{bmatrix} = \begin{bmatrix} a - 2bR^* - cF^* - \lambda U^* & -\lambda R^* & -cR^* \\ \lambda U^* & \lambda R^* - dF^* - e & -dU^* \\ cF^* & dF^* & cR^* + dU^* - f \end{bmatrix}$$

$$= \begin{bmatrix} -bR^* & -\lambda R^* & -cR^* \\ \lambda U^* & 0 & -dU^* \\ cF^* & dF^* & 0 \end{bmatrix} \quad \text{[using (5.3.3), (5.3.4) and (5.3.5)]}$$

The characteristic equation is

$$\begin{vmatrix} -bR^* - \mu & -\lambda R^* & -cR^* \\ \lambda U^* & 0 - \mu & -dU^* \\ cF^* & dF^* & 0 - \mu \end{vmatrix} = 0$$

$$\implies (-bR^* - \mu)(\mu^2 + d^2F^*U^*) + \lambda R^*(-\lambda U^*\mu + cdU^*F^*) - cR^*(\lambda U^*dF^* + cF^*\mu) = 0$$

$$\implies (bR^* + \mu)(\mu^2 + d^2F^*U^*) + \lambda R^*(\lambda U^*\mu - cdU^*F^*) + cR^*(\lambda U^*dF^* + cF^*\mu) = 0$$

$$\implies \mu^3 + bR^*\mu^2 + d^2F^*U^*\mu + bd^2R^*F^*U^* + (\lambda^2R^*U^* + c^2R^*F^*)\mu = 0$$

$$\implies \mu^3 + \mu^2(bR^*) + \mu(d^2F^*U^* + \lambda^2R^*U^* + c^2R^*F^*) + bd^2R^*F^*U^* = 0$$

Comparing the equation with $\mu^3 + a_1\mu^2 + a_2\mu + a_3 = 0$, we have $a_1 = bR^* > 0$, $a_3 = d^2F^*U^*bR^* > 0$ and

$$a_1a_2 - a_3 = bR^*(d^2F^*U^* + \lambda^2R^*U^* + c^2R^*F^*) - bd^2R^*F^*U^* = bR^*(\lambda^2R^*U^* + c^2R^*F^*) > 0$$

Hence by Routh Hurwitz criteria, the characteristic equation has roots with negative real parts and so the interior equilibrium (R^*, U^*, F^*) is stable.

Stability for (*R,U,0*)

$$J(R, U, 0) = \begin{bmatrix} a - 2bR - \lambda U & -\lambda R & -cR \\ \lambda U & \lambda R - e & -dU \\ 0 & 0 & cR + dU - f \end{bmatrix}$$

$$\lambda_1 = cR + dU - f$$

$$(a - 2bR - \lambda U - \gamma)(\lambda R - e - \gamma) + \lambda^2 RU = 0$$

$$\implies \gamma^2 + \gamma(2bR + \lambda U - a + e - \lambda R) + (\lambda R - e)(a - 2bR - \lambda U) + \lambda^2 RU = 0$$

$$\implies \gamma^2 + \gamma(2bR + \lambda U - a + e - \lambda R) + (\lambda R - e)(a - 2bR) - \lambda^2 RU + \lambda eU + \lambda^2 RU = 0$$

$$\implies \gamma^2 + \gamma(2bR + \lambda U - a + e - \lambda R) + (\lambda R - e)(a - 2bR) + \lambda eU = 0$$

$$\left.\begin{array}{c} cR + dU - f < 0 \\ 2bR + \lambda U - a + e - \lambda R > 0 \\ (\lambda R - e)(a - 2bR) + \lambda eU > 0 \end{array}\right\} \qquad (5.3.6)$$

If the three conditions in (5.3.6) hold, then the roots will have negative real parts and so $(R, U, 0)$ will be stable. Otherwise, it is unstable.

5.4 Horizontal and Vertical Transmission

Horizontal: From one individual to another in the same generation. This kind of transmission can occur by direct contact (licking, touching, biting) or indirect contact (by vectors).

Vertical: Passing a disease vertically from parent to offspring. Typically mother transmits the disease by means of bodily fluid, breast milk.

5.5 Ratio dependent prey predator model

The classical model, prey dependent prey predator model, Michaelis-Menten (chemical-kinetics) functional response and logistic growth in the prey population.

$$\left.\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\frac{mN}{a+N}P \\ \frac{dP}{dt} &= \frac{mNP}{a+N}-dP \end{aligned}\right\} \tag{5.5.1}$$

$N(t)$ and $P(t)$ be the concentration of prey and predator population respectively.

Based on the ratio dependent theory, system (5.5.1) can be written as

$$\left.\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\frac{mNP}{N+aP} \\ \frac{dP}{dt} &= \frac{\alpha mNP}{N+aP}-dP \end{aligned}\right\} \tag{5.5.2}$$

α is conversion rate and $N+aP$ is prey-predator dependence concentration.

For equilibrium,

$$\begin{aligned} \frac{dN}{dt}=0 &\implies rN\left(1-\frac{N}{K}\right)-\frac{mNP}{N+aP}=0 \\ &\implies r\left(1-\frac{N}{K}\right)-\frac{mP}{N+aP}=0 \\ &\implies r\left(1-\frac{N}{K}\right)=\frac{mP}{N+aP} \end{aligned} \tag{5.5.3}$$

$$\begin{aligned} \frac{dP}{dt}=0 &\implies \frac{\alpha mNP}{N+aP}-dP=0 \\ &\implies \alpha\left(\frac{mP}{N+aP}\right)N=dP \\ &\implies \alpha r\left(1-\frac{N}{K}\right)N=dP \end{aligned} \tag{5.5.4}$$

Dividing (5.5.4) by (5.5.3) we get

$$\begin{aligned} & \alpha N=\frac{dP}{\frac{mP}{N+aP}} \\ \implies & \alpha N=\frac{d(N+aP)}{m} \\ \implies & \alpha mN=dN+adP \\ \implies & \boxed{P=\left(\frac{\alpha m-d}{ad}\right)N} \end{aligned}$$

Putting this value of P in (5.5.4) we get

$$\begin{aligned}
&\alpha r\left(1-\frac{N}{K}\right)N = d\left(\frac{\alpha m-d}{ad}\right)N \\
\implies\ &\alpha r\left(1-\frac{N}{K}\right) = \frac{\alpha m-d}{a} \\
\implies\ &1-\frac{N}{K} = \frac{\alpha m-d}{a\alpha r} \\
\implies\ &\frac{N}{K} = 1-\frac{\alpha m-d}{a\alpha r} \\
\implies\ &N = K\left(1-\frac{\alpha m-d}{a\alpha r}\right) \\
\therefore\ &\boxed{N^* = K\left(1-\frac{\alpha m-d}{a\alpha r}\right)}
\end{aligned}$$

and $P^* = \left(\frac{\alpha m-d}{ad}\right)N^*$ gives

$$\boxed{P^* = \left(\frac{K(\alpha m-d)}{ad}\right)\left(1-\frac{\alpha m-d}{a\alpha r}\right)}$$

(N^*, P^*) as determined above gives the interior equilibrium.

Stability

$$\begin{aligned}
\frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\frac{mNP}{N+aP} \;=\; f_1(N,P)\\
\frac{dP}{dt} &= \frac{\alpha mNP}{N+aP}-dP \;=\; f_2(N,P)
\end{aligned}$$

Consider the variational matrix

$$\begin{aligned}
&\begin{bmatrix} r\left(1-\frac{N^*}{K}\right)+N^*\left(-\frac{r}{K}\right)-\frac{mP^*}{N^*+aP^*}+\frac{mN^*P^*}{(N^*+aP^*)^2} & -\frac{mN^*}{N^*+aP^*}-\frac{amN^*P^*}{(N^*+aP^*)^2} \\ \frac{\alpha mP^*}{N^*+aP^*}-\frac{\alpha mN^*P^*}{(N^*+aP^*)^2} & \frac{\alpha mN^*}{N^*+aP^*}-\frac{a\alpha mN^*P^*}{(N^*+aP^*)^2}-d \end{bmatrix}\\
=\ &\begin{bmatrix} -\frac{rN^*}{K}+\frac{mN^*P^*}{(N^*+aP^*)^2} & -\frac{m{N^*}^2}{(N^*+aP^*)^2} \\ \frac{\alpha am{P^*}^2}{(N^*+aP^*)^2} & \frac{\alpha m{N^*}^2}{(N^*+aP^*)^2}-d \end{bmatrix}\\
=\ &\begin{bmatrix} AN^*P^*-\frac{rN^*}{K} & -A{N^*}^2 \\ a\alpha{P^*}^2A & \alpha{N^*}^2A-d \end{bmatrix} \qquad \left[\text{let } A = \frac{m}{(N^*+aP^*)^2}\right]
\end{aligned}$$

The characteristic equation is

$$\begin{vmatrix} AN^*P^* - \frac{rN^*}{K} - \lambda & -AN^{*^2} \\ a\alpha P^{*^2}A & \alpha N^{*^2}A - d - \lambda \end{vmatrix} = 0$$

$$\implies \left(AN^*P^* - \frac{rN^*}{K} - \lambda\right)\left(\alpha N^{*^2}A - d - \lambda\right) + a\alpha A^2 P^{*^2} N^{*^2} = 0$$

$$\implies \alpha A^2 N^{*^3} P^* - dAN^*P^* - \frac{r\alpha N^{*^3}A}{K} + \frac{rdN^*}{K} + \lambda\left(\frac{rN^*}{K} - AN^*P^* - \alpha N^{*^2}A + d\right) + \lambda^2 + a\alpha A^2 P^{*^2} N^{*^2} = 0$$

$$\implies \lambda^2 + \lambda\left(\frac{rN^*}{K} - AN^*P^* - \alpha N^{*^2}A + d\right) + \left(\alpha A^2 N^{*^3} P^* - dAN^*P^* - \frac{r\alpha N^{*^3}A}{K} + \frac{rdN^*}{K} + a\alpha A^2 P^{*^2} N^{*^2}\right) = 0$$

The equilibrium point (N^*, P^*) is stable (that is, the roots of the characteristic equation have negative real parts) if the following conditions are satisfied:

$$\frac{rN^*}{K} - AN^*P^* - \alpha N^{*^2}A + d > 0$$

$$\implies \left(\frac{rN^*}{K} + d\right) > \left(AN^*P^* + \alpha N^{*^2}A\right)$$

and

$$\alpha A^2 N^{*^3} P^* - dAN^*P^* - \frac{r\alpha N^{*^3}A}{K} + \frac{rdN^*}{K} + a\alpha A^2 P^{*^2} N^{*^2} > 0$$

$$\implies \left(\alpha A^2 N^{*^3} P^* + \frac{rdN^*}{K} + a\alpha A^2 P^{*^2} N^{*^2}\right) > \left(dAN^*P^* + \frac{r\alpha N^{*^3}A}{K}\right)$$

Units 7 & 8

7.1 Delay-Differential Equation

7.1.1 Delay-Population Model

Let us assume that, the rate at which a population is growing at time t depends on the magnitude of the population at the same time. For example, Mathusian population growth

$$\frac{dN(t)}{dt} = rN(t), \quad N(0) = N_0. \tag{7.1.1}$$

Now, if we know that, present growth rate depends not on the present magnitude but on the magnitude of earlier time. For example, the present growth rate of a family of flies depends not on the number of flies right now but rather on the number of flies laying a certain number of eggs a week ago. In this case,

$$\frac{dN(t)}{dt} = rN(t-\tau) \tag{7.1.2}$$

where τ is the average incubation period of the egg is a time delay or time lag.

7.1.2 Types of Delay-differential equation

Discrete delay-differential equation

$$\frac{dx}{dt} = r_1 x(t) + r_2 x(t-\tau) \tag{7.1.3}$$

where r_1, r_2 are constants.

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{x(t-\tau)}{x^*}\right]$$

when x^* is the carrying capacity.

Continuous or Distributed delay-differential equation

For distributed time delay, the logic equation becomes

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{1}{x^*}\int_{-\infty}^{t} K(t-\tau)x(\tau)d\tau\right] \tag{7.1.4}$$

which is the example of integro-differential equation. The function $K(t-\tau)$ is called delay-kernel.

7.1.3 Discrete time delay model

Let us consider a model equation of the form

$$\frac{dx(t)}{dt} = rx(t)g(x(t-\tau)) \tag{7.1.5}$$

where $g(x(t-\tau)) = 1 - \dfrac{x(t-\tau)}{x^*}$. An equilibrium point of equation (7.1.5) is the value of x^* such that

$$\begin{aligned} & x(t)g(x(t-\tau)) = 0 \\ \Rightarrow\quad & x^*g(x^*) = 0. \end{aligned}$$

Therefore, the logistic delay equation has two equilibrium points, $x = 0$ and $x = x^*$.

Theorem 7.1.1. If all the solutions of the equation

$$\frac{du(t)}{dt} = r\left[g(x^*)u(t) + x^*g'(x^*)u(t-\tau)\right]$$

tends to zero as $t \to \infty$, then every solution $x(t)$ of equation (7.1.5) with $|x(t) - x^*|$ sufficiently small tend to the equilibrium point x^* as $t \to \infty$.

7.1.4 Asymptotic Stability

For the differential equation

$$\frac{dx(t)}{dt} = xg(x) \tag{7.1.6}$$

which in this case $\tau = 0$ and the equilibrium point x^* is said to be asymptotically stable if and only if

$$\frac{d}{dx}\left(xg(x)\right)\Big|_{x=x^*} = g\left(x^*\right) + x^*g'\left(x^*\right) < 0. \tag{7.1.7}$$

Therefore, the equilibrium $x^* = 0$ is asymptotically stable if $g(0) < 0$ and $x^* > 0$ is asymptotically stable if $g'(x^*) < 0$.

a. For $x^* = 0$, $\dfrac{du}{dt} = rg(0)u(t)$. Since, $g(0) = 1 > 0$. Therefore, the equilibrium point $x^* = 0$ is unstable.

b. For equilibrium $x^* > 0$,

$$\begin{aligned} g(x^*) &= 0 \\ u'(t) &= rx^*g'(x^*)u(t-\tau) \\ &= rbu(t-\tau). \quad [\text{where, } b = x^*g'(x^*)] \end{aligned} \tag{7.1.8}$$

In order to determine whether all the solutions of the linear differential equation tends to zero as $t \to \infty$, we take a solution of the form

$$u(t) = c\,e^{\lambda t} \quad (c \text{ is a constant}).$$

Putting this in (7.1.8),

$$\begin{aligned} & c\lambda\,e^{\lambda t} = rbc\,e^{\lambda(t-\tau)} \\ \Rightarrow\quad & \lambda\,e^{\lambda t} = rb\,e^{\lambda t}\cdot e^{-\lambda\tau} \\ \Rightarrow\quad & \lambda = rb\,e^{-\lambda\tau}\,. \end{aligned} \tag{7.1.9}$$

This is transcendental equation for τ having infinitely many roots.

We assume that, if all the characteristic roots of equation (7.1.9) have negative real parts, then all solutions of the equation (7.1.8) tend to zero as $t \to \infty$. In the delay case with $\tau > 0$, it is possible to show that the condition that all roots of the characteristic equation (7.1.9) have negative real part is

$$0 < -rb\tau < \frac{\pi}{2}. \tag{7.1.10}$$

The condition (7.1.10) implies that $b < 0$ and in addition to that, time lag (delay) not to be too large. Combining the analysis with the above theorem, we see that, an equilibrium point $x^* > 0$ of $\frac{dx(t)}{dt} = rx(t)g(x(t-\tau))$ is asymptotically stable if $0 < -rx^*g'(x^*)\tau < \frac{\pi}{2}$

Example 7.1.2. 1. For delay logistic model

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{x(t-\tau)}{x^*}\right].$$

Show that the stability condition becomes $0 < r\tau < \frac{\pi}{2}$.

2. Show that the equilibrium point $x^* = k$ of the delay equation $\frac{dx(t)}{dt} = rx(t)\log\left(\frac{k}{x(t-\tau)}\right)$ is asymptotically stable if $0 < r\tau < \pi/2$.

Solution. 1.

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{x(t-\tau)}{x^*}\right].$$

For logistic model, $x = 0$ and $x = x^*$ are two equilibrium points. Let

$$g(x(t-\tau)) = 1 - \frac{x(t-\tau)}{x^*}$$
$$g'(x(t-\tau))\big|_{x=x^*} = -\frac{1}{x^*}.$$

Now,

$$x^*g'(x^*) = x^*\left(-\frac{1}{x^*}\right) = -1.$$

Therefore, the stability condition becomes,

$$0 < -rb\tau < \frac{\pi}{2} \Rightarrow 0 < r\tau < \frac{\pi}{2} \;\; [b = -1].$$

2.

$$\frac{dx(t)}{dt} = rx(t)\log\left(\frac{k}{x(t-\tau)}\right).$$

Here, $0, k$ are two equilibrium points of the Gompertz growth model.

$$\begin{aligned}
g(x(t-\tau)) &= \log\left[\frac{k}{x(t-\tau)}\right] \\
g'(x(t-\tau)) &= \frac{1}{\frac{1}{x(t-\tau)}}\left(-\frac{k}{(x(t-\tau))^2}\right) \\
&= -\frac{x(t-\tau)}{k}\frac{k}{(x(t-\tau))^2} \\
&= -\frac{1}{x(t-\tau)} \\
g'(k) &= -\frac{1}{k} \text{ [Here, } x^* = k]. \\
\text{Therefore, } kg'(k) &= -k \cdot \frac{1}{k} = -1.
\end{aligned}$$

Therefore, $b = -1$. The stability condition becomes

$$0 < -rb\tau < \frac{\pi}{2} \Rightarrow 0 < r\tau < \frac{\pi}{2}.$$

■

7.2 Distributed Delay

For discrete time delay, the logistic equation is

$$\frac{dx(t)}{dt} = rx(t)g(x(t-\tau)) \tag{7.2.1}$$

where $gx(t) = 1 - \frac{x(t)}{x^*}$. In case of distributed delay, this equation can be generalised as

$$\frac{dx(t)}{dt} = x(t)\int_0^\infty g(x(t-\tau))p(s)ds. \tag{7.2.2}$$

Here, $p(s)ds$ represents the probability of a delay between s and $s + ds$. Therefore,

$$\int_0^\infty p(s)ds = 1 \text{ [since } \tau > 0].$$

An equilibrium point of the integro-differential equation

$$\frac{dx(t)}{dt} = x(t)\int_0^\infty g(x(t-s))p(s)ds$$

is a value x^* such that,

$$x^*\int_0^\infty g(x^*)p(s)ds = 0 \quad \text{or, } x^*g(x^*) = 0.$$

We see that $x^* = 0$ is an equilibrium point and the equilibrium $x^* > 0$ is given by $g(x^*) = 0$.

7.2.1 Linearization about an equilibrium point

To linearize

$$\frac{dx(t)}{dt} = x(t)\int_0^\infty g(x(t-s))p(s)ds \tag{7.2.3}$$

about an equilibrium point x^*, we put $u(t) = x(t) - x^*$. We have,

$$\begin{aligned}
\frac{du(t)}{dt} &= (u(t)+x^*)\int_0^\infty g(x^*+u(t-s))p(s)ds \\
&= (u(t)+x^*)\int_0^\infty \left[g(x^*)+u(t-s)g'(x^*)+\frac{(u(t-s))^2}{2!}g''(x^*)+\dots\right]p(s)ds \\
&= (u(t)+x^*)\left[g(x^*)+g'(x^*)\int_0^\infty u(t-s)p(s)ds+\dots\right] \\
&= x^*g(x^*)+g(x^*)u(t)+x^*g'(x^*)\int_0^\infty u(t-s)p(s)ds+\dots
\end{aligned} \tag{7.2.4}$$

We are now in a state to study the integro-differential equation of the form

$$\frac{du(t)}{dt} = au(t)+b\int_0^\infty u(t-s)p(s)ds \tag{7.2.5}$$

where $a = g(x^*),\ b = x^*g'(x^*),\ p(s) \geq 0$ for $0 < s < \infty$.

$$\int_0^\infty p(s)ds = 1.$$

To study the behaviour of the solution of equation (7.2.5), for a specific kernel $p(s)$, we take the solution as

$$\begin{aligned}
& u(t) = c\,\mathrm{e}^{\lambda t} \qquad (7.2.6)\\
& c\lambda\,\mathrm{e}^{\lambda t} = ac\,\mathrm{e}^{\lambda t} + bc\,\mathrm{e}^{\lambda t}\int_0^\infty \mathrm{e}^{-\lambda s}\,p(s)ds \\
\Rightarrow\ & \lambda = a+b\int_0^\infty \mathrm{e}^{-\lambda s}\,p(s)ds \\
\Rightarrow\ & \lambda = a+b\cdot \text{Laplace Transform of } p(s) \\
\Rightarrow\ & \lambda = a+bL(p(s)) \\
\Rightarrow\ & \lambda = a+bF(\lambda), \qquad (7.2.7)
\end{aligned}$$

where $F(\lambda)$ is the Laplace transform of $p(s)$. We shall consider two specific choices of p.

$$\int_0^\infty p(s)ds = 1,\ \int_0^\infty sp(s)ds = T \text{ (Average delay)}.$$

We shall make use of the following results.

1. $\displaystyle\int_0^\infty \mathrm{e}^{-\alpha s}\,ds = \frac{1}{\alpha};$
2. $\displaystyle\int_0^\infty s\,\mathrm{e}^{-\alpha s}\,ds = -\frac{1}{\alpha^2};$
3. $\displaystyle\int_0^\infty s^2\,\mathrm{e}^{-\alpha s}\,ds = \frac{2}{\alpha^3}.$

Let us take $p_1(s) = \mathrm{e}^{-\frac{2s}{T}\left(\frac{4s}{T^2}\right)}$.

Here, $p_1(0) = 0$, $p_1(s)$ has the maximum value at $s = \dfrac{T}{2}$. Now,

$$
\begin{aligned}
L(p_1(s)) &= \int_0^\infty \mathrm{e}^{-\lambda s} p_1(s) ds \\
&= \int_0^\infty \mathrm{e}^{-\lambda s} \cdot \mathrm{e}^{-\frac{2s}{T}\left(\frac{4s}{T^2}\right)} ds \\
&= \frac{4}{T^2} \int_0^\infty \mathrm{e}^{-\left(\lambda+\frac{2}{T}\right)s} \cdot s ds \\
&= \frac{4}{T^2}\left(\left[-s\frac{\mathrm{e}^{-\left(\lambda+\frac{2}{T}\right)s}}{\left(\lambda+\frac{2}{T}\right)}\right]_0^\infty - \int_0^\infty \left\{\frac{d}{ds}(s)\int \mathrm{e}^{-\left(\lambda+\frac{2}{T}\right)s} ds\right\} ds\right) \quad \left[\left(\lambda+\frac{2}{T}\right)s = p\right] \\
&= -\frac{4}{T^2}\int_0^\infty -\frac{\mathrm{e}^{-\left(\lambda+\frac{2}{T}\right)s}}{\left(\lambda+\frac{2}{T}\right)} ds \\
&= \frac{4}{T^2} \cdot \frac{1}{\left(\lambda+\frac{2}{T}\right)} \int_0^\infty \mathrm{e}^{-\left(\lambda+\frac{2}{T}\right)s} ds \\
&= -\frac{4}{T^2} \cdot \frac{1}{\left(\lambda+\frac{2}{T}\right)^2}\left[\frac{1}{\mathrm{e}^{\left(\lambda+\frac{2}{T}\right)s}}\right]_0^\infty \\
&= -\frac{4}{T^2} \cdot \frac{1}{\left(\lambda+\frac{2}{T}\right)^2}[0-1] \\
&= \frac{4}{T^2} \cdot \frac{1}{\left(\lambda+\frac{2}{T}\right)^2} \\
&= \frac{4}{\lambda^2 T^2 + 4\lambda T + 4}.
\end{aligned}
$$

Thus, the characteristic equation is

$$
\begin{aligned}
& a + \frac{4b}{\lambda^2 T^2 + 4\lambda T + 4} = \lambda \\
\Rightarrow\ & a(\lambda^2 T^2 + 4\lambda T + 4) + 4b = \lambda(\lambda^2 T^2 + 4\lambda T + 4) \\
\Rightarrow\ & \lambda^3 + \left(\frac{4T - aT^2}{T^2}\right)\lambda^2 + \frac{4 - 4aT}{T^2}\lambda - \frac{4(a+b)}{T^2} = 0.
\end{aligned} \tag{7.2.8}
$$

By Routh Hurwitz Criterion, all the roots of the above equation have negative real part,

$$\frac{4T - aT^2}{T^2} > 0, \quad -\frac{4(a+b)}{T^2} > 0,$$

and

$$\frac{4T - aT^2}{T^2}\frac{4 - 4aT}{T^2} + \frac{4(a+b)}{T^2} > 0.$$

The stability conditions are,

$$a + b < 0, \ \ aT < 4, \quad -bT < (2 - aT)^2. \tag{7.2.9}$$

For the equation,

$$\frac{du(t)}{dt} = au(t) + b\int_0^\infty u(t-s)p(s)ds.$$
$$a \equiv g(x^*), \ \ b \equiv x^* g'(x^*).$$

If the equilibrium is,

1. If $x^* = 0, \;\; b = 0$, then the condition of stability reduces to $a < 0 \Rightarrow g(x^*) < 0$.
2. If $x^* > 0$, then $a = 0$. Since, the equilibrium point satisfies the condition $g(x^*) = 0$. In this case, the stability condition reduces to
$$0 < -\underbrace{x^* g'(x^*)}_{\text{b}} T < 4.$$

Example 7.2.1. $p_2(s) = \dfrac{1}{T}\,\mathrm{e}^{-s/T}$, find the stability condition. $(T > 0)$.

Solution. Now,

$$\begin{aligned} L(p_2(s)) &= \int_0^\infty \mathrm{e}^{-\lambda s}\, p_2(s) ds \\ &= \int_0^\infty \mathrm{e}^{-\lambda s} \cdot \frac{1}{T}\,\mathrm{e}^{-s/T}\, ds \\ &= \frac{1}{T}\int_0^\infty \mathrm{e}^{-\left(\lambda + \frac{1}{T}\right)s}\, ds \\ &= -\frac{1}{T}\left[\frac{\mathrm{e}^{-\left(\lambda+\frac{1}{T}\right)s}}{\left(\lambda+\frac{1}{T}\right)}\right]_0^\infty \\ &= -\frac{1}{T\left(\lambda+\frac{1}{T}\right)}\left[\mathrm{e}^{-\left(\lambda+\frac{1}{T}\right)s}\right]_0^\infty \\ &= -\frac{1}{T\left(\lambda+\frac{1}{T}\right)}[0-1] \\ &= \frac{1}{T\left(\lambda+\frac{1}{T}\right)} = \frac{1}{(\lambda T + 1)}. \end{aligned}$$

The characteristic equation is

$$\begin{aligned} & \lambda = a + bF(\lambda) \\ \Rightarrow\;& \lambda = a + b \cdot \frac{1}{\lambda T + 1} \\ \Rightarrow\;& \lambda = \frac{a(\lambda T + 1) + b}{\lambda T + 1} \\ \Rightarrow\;& \lambda(\lambda T + 1) = a(\lambda T + 1) + b \\ \Rightarrow\;& \lambda^2 T + \lambda = a\lambda T + a + b \\ \Rightarrow\;& \lambda^2 T + (1 - aT)\lambda - (a + b) = 0 \\ \Rightarrow\;& \lambda^2 + \frac{(1 - aT)}{T}\lambda - \frac{a+b}{T} = 0. \end{aligned} \tag{7.2.10}$$

$$\left[\lambda = \frac{-\left(\frac{1-aT}{T}\right) \pm \sqrt{\frac{(1-aT)^2}{T^2} + 4\frac{a+b}{T}}}{2} = \frac{\left(\frac{aT-1}{T}\right) \pm \frac{1}{T}\sqrt{(1-aT)^2 + 4T(a+b)}}{2}\right]$$

For stability condition of (7.2.10),

$$\frac{1 - aT}{T} > 0 \;\text{ and }\; -\left(\frac{a+b}{T}\right) > 0 \Rightarrow \frac{a+b}{T} < 0.$$

Since $T > 0$, then

$$aT < 1 \;\text{ and }\; a + b < 0.$$

These are the stability conditions. ■

For the equation,

$$\frac{du(t)}{dt} = au(t) + b\int_0^{\infty} u(t-s)p(s)ds.$$

Here, $a = g(x^*)$ and $b = x^*g'(x^*)$.

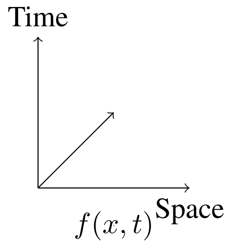
If

1. $x^* = 0$, then $g'(x^*) \cdot x^* = b = 0$, then the stability reduces to $a < 0 \Rightarrow g(x^*) < 0$.
2. $x^* > 0$, then $a = 0$. Since the equilibrium point satisfies the condition $g(x^*) = 0$, then the stability condition reduces to $b < 0 \Rightarrow x^*g'(x^*) < 0$.

Units 9 & 10

9.1 Spatial Model

Let us suppose, we have a population that inhibits a patch of length L $(0 \leq x \leq L)$ that does not grow and is subject to simple diffusion.



$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}. \tag{9.1.1}$$

We have, the initial condition

$$n(x,0) = n_0(s). \tag{9.1.2}$$

Let us assume that, we have homogeneous Dirichlet boundary condition

$$n(0,t) = 0 \tag{9.1.3}$$
$$n(L,t) = 0. \tag{9.1.4}$$

To solve equation (9.1.1) we put,

$$n(x,t) = S(x)T(t) \rightarrow \text{Characteristics} \tag{9.1.5}$$

Putting (9.1.5) in (9.1.1), we get,

$$\frac{1}{D}\frac{\dot{T}}{T} = \frac{S''}{S} \quad \left[S'' = \frac{d^2S}{dt^2}, \; \dot{T} = \frac{dT}{dt}\right] \tag{9.1.6}$$

Since, space and time are independent variable. Therefore, these two can be equal if both of them equals to a constant.

$$\frac{1}{D}\frac{\dot{T}}{T} = \frac{S''}{S} = -\lambda \; (\text{constant}). \tag{9.1.7}$$

This implies

$$\dot{T} = -DT\lambda \quad S'' = -S\lambda.$$

Solving, we get

$$\begin{aligned}T(t) &= C\,\mathrm{e}^{-D\lambda t}\\ S(x) &= a\sin(\sqrt{\lambda}x)+b\cos(\sqrt{\lambda}x)\\ n(x,t) &= (a\sin(\sqrt{\lambda}x)+b\cos(\sqrt{\lambda}x))C\,\mathrm{e}^{-D\lambda t}\\ &= \mathrm{e}^{-D\lambda t}\left[A\sin(\sqrt{\lambda}x)+B\cos(\sqrt{\lambda}x)\right].\end{aligned} \tag{9.1.8}$$

By using the boundary condition,

$$\begin{aligned}n(0,t)=0 &\;\Rightarrow\; B=0\\ n(L,t)=0 &\;\Rightarrow\; \mathrm{e}^{-D\lambda t}\left[A\sin(\sqrt{\lambda}L)\right]=0\\ &\;\Rightarrow\; A\sin(\sqrt{\lambda}L)=0\\ &\;\Rightarrow\; \sin(\sqrt{\lambda}L)=0\;\;[A\neq 0]\\ &\;\Rightarrow\; \sin(\sqrt{\lambda}L)=0=\sin k\pi\\ &\;\Rightarrow\; \lambda=\frac{k^2\pi^2}{L^2},\;\; k \text{ is an integer.}\end{aligned}$$

$$\begin{aligned}n_k(x,t) &= \mathrm{e}^{-D\lambda t}\left(A_k\sin\frac{k\pi x}{L}\right)\\ &= \mathrm{e}^{-Dt\frac{k^2\pi^2}{L}}\left(A_k\sin\frac{k\pi x}{L}\right).\end{aligned} \tag{9.1.9}$$

In more general form, the solution can be written as

$$n(x,t)=\sum_{k=1}^{\infty}A_k\,\mathrm{e}^{-Dt\frac{k^2\pi^2}{L}}\sin\left(\frac{k\pi x}{L}\right). \tag{9.1.10}$$

Now, using the initial condition $n(x,0)=n_0$,

$$n_0=\sum_{k=1}^{\infty}A_k\sin\left(\frac{k\pi x}{L}\right). \tag{9.1.11}$$

Our initial condition can be written as Fourier 'sine' series with the coefficient in the sine series being the coefficients of our solution.

These coefficients can also be taken as coordinates with the various sine functions as basis vectors. We can now use the orthogonality property of this basis vector

$$\begin{aligned}\int_0^L \sin\left(\frac{j\pi x}{L}\right)\cdot\sin\left(\frac{k\pi x}{L}\right)dx &= 0,\;\; j\neq k\\ &= \frac{L}{2},\;\; j=k.\end{aligned}$$

By using orthogonality property,

$$\int_0^L n_0(x)\sin\left(\frac{k\pi x}{L}\right)dx=\frac{L}{2}A_k$$

$$\text{or,}\quad A_k=\frac{2}{L}\int_0^L n_0(x)\sin\left(\frac{k\pi x}{L}\right)dx.$$

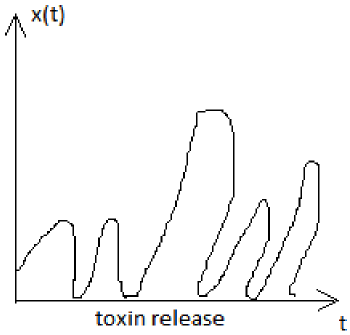
Therefore,

$$n_0(x) = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi x}{L}, \;\text{ where }\; A_k = \frac{2}{L}\int_0^L n_0(x)\sin\left(\frac{k\pi x}{L}\right)dx.$$

We now introduce the addition of growth into our system.

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}.$$

Kierstead and Slobodkin were in phytoplankton bloom. For example, red tide out breaks. They assumed



simple exponential growth, diffusion and homogeneous Dirichlet condition, then the system becomes,

$$\frac{\partial n}{\partial t} = rn + D\frac{\partial^2 n}{\partial x^2} \tag{9.1.12}$$

$$\begin{aligned} \text{with boundary condition } n(0,t) &= n(L,t) = 0 \\ \text{initial condition } n(x,0) &= n_0(x). \end{aligned}$$

It is possible to reduce this equation to the previously solved problem by,

$$n(x,t) = \mathrm{e}^{rt}\,u(x,t)$$

$$\begin{aligned} \frac{\partial n}{\partial t} = r\,\mathrm{e}^{rt}\,u(x,t) + \mathrm{e}^{rt}\,\frac{\partial u}{\partial t} \quad &= \quad r\,\mathrm{e}^{rt}\,u(x,t) + D\,\mathrm{e}^{rt}\,\frac{\partial^2 u}{\partial x^2} \\ &\Rightarrow \quad \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \end{aligned} \tag{9.1.13}$$

$$\begin{aligned} &u(0,t) = 0 = u(L,t) \\ &u(x,0) = u_0(x) \;\; \text{[To prove previous problem]}. \end{aligned}$$

The solution of this equation,

$$u(x,t) \quad = \quad \sum_{k=1}^{\infty} A_k\,\mathrm{e}^{-\frac{Dtk^2\pi^2}{L^2}}\sin\left(\frac{k\pi}{L}\right).$$

$$\text{Therefore,} \qquad n(x,t) = \sum_{k=1}^{\infty} A_k\,\mathrm{e}^{\left(r-\frac{Dtk^2\pi^2}{L^2}\right)t}\sin\left(\frac{k\pi}{L}\right). \tag{9.1.14}$$

This population grow if

$$r - \frac{Dtk^2\pi^2}{L^2} > 0 \Rightarrow r > \frac{Dtk^2\pi^2}{L^2} \Rightarrow L > k\pi\sqrt{\frac{D}{r}}$$

For $k = 1$

$$L > \pi\sqrt{\frac{D}{r}}. \tag{9.1.15}$$

The critical length L of the equation (9.1.15) is referred to as 'Kiss size'. If the patch is shorter than this length, the population collapses and if it is longer than this length, a bloom occurs.

Let us consider the non-linear model

$$\frac{\partial n}{\partial t} = rn\left(1 - \frac{n}{k}\right) + D\frac{\partial^2 n}{\partial x^2}, \quad 0 < x < L \tag{9.1.16}$$

$$\begin{aligned} &\text{with boundary condition } n(0,t) = 0 = n(L,t) \\ &\text{initial condition, } n(x,0) = n_x(0). \end{aligned} \tag{9.1.17}$$

Fick's law Equation (9.1.16) contains logistic growth and simple Fickian. It is called Fisher's equation.

Equation (9.1.16) contains three parameters $r,\ k,\ D$. Let

$$u(x,t) = \frac{n(x,t)}{k} \tag{9.1.18}$$

Then (9.1.16) becomes,

$$\frac{\partial u}{\partial t} = ru(1-u) + D\frac{\partial^2 u}{\partial x^2} \quad 0 < x < L \tag{9.1.19}$$

$$\begin{aligned} u(0,t) &= u(L,t) = 0 \\ u(x,0) &= u_0(x). \end{aligned} \tag{9.1.20}$$

The steady-state solution

$$\begin{aligned} &ru(1-u) + Du'' = 0 \quad \left[\frac{\partial u}{\partial t} = 0\right] \\ &u(0) = u(L) = 0 \end{aligned} \tag{9.1.21}$$

We write equation (9.1.21) as

$$\begin{aligned} u' &= v \\ v' &= -\frac{r}{D}u(1-u). \end{aligned} \tag{9.1.22}$$

For equilibrium, $u' = 0, v' = 0$.

$$\begin{aligned} &u' \ \text{ gives } v = 0 \\ &v' \Rightarrow u = 0,\ u = 1. \end{aligned}$$

Therefore, the equilibrium points are $(0,0)$ and $(1,0)$. The variational matrix

$$\begin{bmatrix} 0 & 1 \\ -\frac{r}{D}(1-2u) & 0 \end{bmatrix}.$$

At $(0,0)$,

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{r}{D}(1-2u) & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + \frac{r}{D}(1-2u) = 0$$

$$\Rightarrow \quad \lambda = \pm\sqrt{\frac{r}{D}}i.$$

At $(1,0)$,

$$\begin{vmatrix} -\lambda & 1 \\ \frac{r}{D} & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{\frac{r}{D}}.$$

Therefore, $(1,0)$ is saddle. For $(0,0)$ it is a centre. However, linearization is unreliable for non-hyperbolic equilibrium point.

Equation (9.1.21) has a first integral, multiplying (9.1.21) by u',

$$Du'u'' + ruu'(1-u) = 0.$$

Integrating with respect to x,

$$D\frac{(u')^2}{2} + r\left(\frac{u^2}{2} - \frac{u^3}{3}\right) = c.$$

This equation can be written as

$$\frac{v^2}{2} + \frac{r}{D}\left(\frac{u^2}{2} - \frac{u^3}{3}\right) = 0. \tag{9.1.23}$$

The phase-portrait is symmetric in $v = u'$.

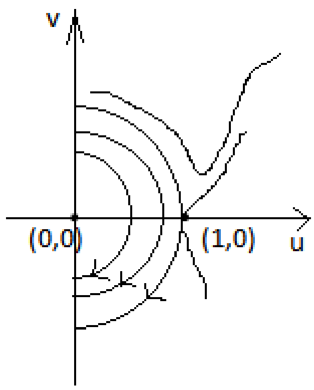


Figure 9.1.1: Phase plane for the steady states of the Fisher's equations

Let us assume that, we have a solution of equation (9.1.21) that satisfies the boundary conditions, equation (9.1.23) may now be written as,

$$\frac{v^2}{2} + \frac{r}{D}\left(\frac{u^2}{2} - \frac{u^3}{3}\right) = \frac{r}{D}F(\mu)$$

where $u = \mu$ when $v = 0$ at $x = \dfrac{L}{2}$.

$$v = \frac{du}{dx} = \sqrt{\frac{2r}{D}\left[F(\mu) - F(u)\right]}, \quad 0 < x < \frac{L}{2} \tag{9.1.24}$$

$$= -\sqrt{\frac{2r}{D}\left[F(\mu) - F(u)\right]}, \quad \frac{L}{2} < x < L. \tag{9.1.25}$$

$$\begin{aligned}
&\text{From (9.1.24)} \quad \sqrt{\frac{D}{2r}}\int_0^{\mu}\frac{du}{\sqrt{F(\mu)-F(u)}} = \int_0^{\frac{L}{2}} dx \\
&\text{From (9.1.25)} \quad \sqrt{\frac{D}{2r}}\int_{\mu}^{0}\frac{du}{\sqrt{F(\mu)-F(u)}} = \int_{\frac{L}{2}}^{L} dx.
\end{aligned} \tag{9.1.26}$$

$$L = \sqrt{\frac{2D}{r}}\int_0^{\mu}\frac{du}{\sqrt{F(\mu)-F(u)}}. \tag{9.1.27}$$

Let us take the substitution $z = \frac{u}{\mu}$.

$$L = \sqrt{\frac{2D}{r}}\int_0^{1}\frac{\mu dz}{\sqrt{F(\mu)-F(\mu z)}}. \tag{9.1.28}$$

Equation (9.1.28) may be considered as an elliptic integral, we can deduce the following facts from (9.1.28).

1. L is an increasing function of μ for $0 \leq \mu \leq 1$.

2. $\lim_{\mu\to 1} L(\mu) \to \infty$.

3. $\lim_{\mu\to 1} L(\mu) = L_c = \pi\sqrt{\frac{D}{r}}$.

Proof. 3.

$$\begin{aligned}
\lim_{\mu\to 1} L(\mu) &= \lim_{\mu\to 1}\sqrt{\frac{2D}{r}}\int_0^1\frac{\mu dz}{\sqrt{F(\mu)-F(\mu z)}} \\
&= \lim_{\mu\to 1}\sqrt{\frac{2D}{r}}\int_0^1\frac{\mu dz}{\sqrt{\frac{\mu^2}{2}-\frac{\mu^3}{3}-\frac{\mu^2 z^2}{2}+\frac{\mu^3 z^3}{3}}} \\
&= \lim_{\mu\to 1}\sqrt{\frac{2D}{r}}\int_0^1\frac{dz}{\sqrt{\frac{1}{2}-\frac{\mu}{3}-\frac{z^2}{2}+\frac{\mu z^3}{3}}} \\
&= \sqrt{\frac{2D}{r}}\sqrt{2}\int_0^1\frac{dz}{\sqrt{1-z^2}} \\
&= 2\sqrt{\frac{D}{r}}[\sin^{-1} z]_0^1 \\
&= 2\sqrt{\frac{D}{r}}\frac{\pi}{2} \\
&= \pi\sqrt{\frac{D}{r}}.
\end{aligned}$$

2.

$$
\begin{aligned}
\lim_{\mu\to 1} L(\mu) &= \lim_{\mu\to 1}\sqrt{\frac{2D}{r}}\int_0^1 \frac{\mu dz}{\sqrt{F(\mu)-F(\mu z)}}\\
&= \lim_{\mu\to 1}\sqrt{\frac{2D}{r}}\int_0^1 \frac{\mu dz}{\sqrt{\frac{\mu^2}{2}-\frac{\mu^3}{3}-\frac{\mu^2 z^2}{2}+\frac{\mu^3 z^3}{3}}}\\
&= \lim_{\mu\to 1}\sqrt{\frac{2D}{r}}\int_0^1 \frac{dz}{\sqrt{\frac{1}{2}-\frac{\mu}{3}-\frac{z^2}{2}+\frac{\mu z^3}{3}}}\\
&= \sqrt{\frac{2D}{r}}\int_0^1 \frac{dz}{\sqrt{\frac{1}{2}-\frac{1}{3}-\frac{z^2}{2}+\frac{z^3}{3}}}\\
&= \sqrt{\frac{2D}{r}}\int_0^1 \frac{dz}{\sqrt{\frac{1}{2}(1-z)(1+z)-\frac{1}{3}(1-z)(1+z+z^2)}}\\
&= \sqrt{\frac{2D}{r}}\int_0^1 \frac{dz}{\sqrt{\frac{1}{6}(1-z)\left[3(1+z)-2(1+z+z^2)\right]}}\\
&= \sqrt{6}\sqrt{\frac{2D}{r}}\int_0^1 \frac{dz}{\sqrt{(1-z)(1+z-2z^2)}}\\
&= \sqrt{6}\sqrt{\frac{2D}{r}}\int_0^1 \frac{dz}{\sqrt{(1-z)(1-z)(1+2z)}}\\
&= \sqrt{6}\sqrt{\frac{2D}{r}}\int_0^1 \frac{dz}{(1-z)\sqrt{(1+2z)}}\\
&= \sqrt{6}\sqrt{\frac{2D}{r}}\int_1^{\sqrt{3}} \frac{pdp}{p\left(1-\frac{p^2-1}{2}\right)} \quad \text{Taking } 1+2z=p^2\\
&= \sqrt{6}\sqrt{\frac{2D}{r}}\int_1^{\sqrt{3}} \frac{2dp}{2-p^2+1}\\
&= 2\sqrt{6}\sqrt{\frac{2D}{r}}\int_1^{\sqrt{3}} \frac{dp}{3-p^2}\\
&= 2\sqrt{6}\sqrt{\frac{2D}{r}}\times\frac{1}{2\sqrt{3}}\left[\log\left|\frac{\sqrt{3}+p}{\sqrt{3}-p}\right|\right]_1^{\sqrt{4}}\\
&= 2\sqrt{\frac{D}{r}}\left[\log\left(\frac{2\sqrt{3}}{0}\right)-\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right]\\
&= 2\sqrt{\frac{D}{r}}\left[\log\infty-\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right]\to\infty.
\end{aligned}
$$

□

9.1.1 Exponential growth and spatial spread in an infinite domain

Let us consider a simple model of exponential population growth with simple Fickian equation

$$\frac{\partial n}{\partial t} = rn + D\frac{\partial^2 n}{\partial x^2} \tag{9.1.29}$$

with initial condition

$$n(x,0) = f(x). \tag{9.1.30}$$

The substitution

$$n(x,t) = \mathrm{e}^{rt}\, u(x,t) \tag{9.1.31}$$

reduces (9.1.29) into

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \tag{9.1.32}$$

$$u(x,0) = f(x).$$

We will assume that initial function $f(x)$ goes to zero and the solution $u(x,t)$ is bounded for large positive or negative values of x.

$$\text{Let } \; u(x,t) = S(x)T(t).$$

$$\frac{1}{D}\frac{\dot{T}}{T} = \frac{S''}{S} = -\lambda = -k^2 \tag{9.1.33}$$

$$S''(x) = k^2 S(x) = 0 \tag{9.1.34}$$

The real solutions of these equations are in terms of sines and cosines. But, for convenience, we will use the complex exponential e^{ikx} and e^{-ikx}. Therefore, there are continuous functions of the form

$$u(x,t,k) = F(k)\,\mathrm{e}^{ikx-k^2Dt} \;\; \text{(Solve)} \tag{9.1.35}$$

To write that equation (9.1.35) satisfies (9.1.32). With this we need an integral

$$u(x,t) = \int_{-\infty}^{\infty} F(k)\,\mathrm{e}^{ikx-k^2Dt}\,dk. \tag{9.1.36}$$

Initial condition $u(x,0) = f(x)$ must satisfy

$$f(x) = \int_{-\infty}^{\infty} K(k)\,\mathrm{e}^{ikx}\,dk. \tag{9.1.37}$$

Equation (9.1.37) can be inverted by using Fourier inversion formula

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-ikx}\,dx. \tag{9.1.38}$$

Equation (9.1.37) and (9.1.38) together form a Fourier transform. Substituting (9.1.38) into (9.1.36)

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\mathrm{e}^{ikx-k^2Dt}\int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-ik\xi}\,d\xi\right]dk.$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{ikx-k^2Dt-ik\xi}\,f(\xi)d\xi dk.$$

Now,

$$\mathrm{e}^{-Dt\left[k^2-\frac{ik(x-\xi)}{Dt}\right]} = \mathrm{e}^{-Dt\left[k-\frac{i(x-\xi)}{2Dt}\right]^2-\frac{(x-\xi)^2}{4D^2t^2}}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ie^{-\frac{(x-\xi)^2}{4Dt}} f(\xi)d\xi$$

where

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-Dt\left[k-\frac{i(x-\xi)}{2Dt}\right]^2} dk.$$

Let $z = k - \dfrac{i(x-\xi)}{2Dt}$. Then

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{e}^{-Dtz^2}\, dz \\ &= \frac{1}{2\sqrt{Dt}}\int_{-\infty}^{\infty} \mathrm{e}^{-p}\, p^{-\frac{1}{2}} dp \;\; [p = Dtz^2] \\ &= \frac{1}{2\sqrt{Dt}}\int_{0}^{\infty} \mathrm{e}^{-p}\, p^{\frac{1}{2}-1} dp \\ &= \frac{1}{\sqrt{Dt}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{Dt}}. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{Dt}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-\xi)^2}{4Dt}} f(\xi)d\xi \\ &= \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-\xi)^2}{4Dt}} f(\xi)d\xi \\ &= \int_{-\infty}^{\infty} g(x-\xi,t)f(\xi)d\xi \end{aligned} \tag{9.1.39}$$

where

$$g(x-\xi,t) = \frac{1}{2\sqrt{\pi Dt}}\mathrm{e}^{-\frac{(x-\xi)^2}{4Dt}}. \tag{9.1.40}$$

Equation (9.1.40) is known as fundamental solution of Heat equation. It describes the evolution of a point releasing heat.

Now, from equation (9.1.29) and substituting (9.1.31) we see that a point source of size n_0 will grow like

$$n(x,t) = \frac{n_0}{2\sqrt{\pi Dt}}\mathrm{e}^{rt-\frac{(x-\xi)^2}{4Dt}}. \tag{9.1.41}$$

Now, let us imagine that the population is detectable when it reaches a certain threshold density n_c. We can solve from (9.1.41)

$$n_c = \frac{n_0}{2\sqrt{\pi Dt}}\mathrm{e}^{rt-\frac{(x-\xi)^2}{4Dt}}.$$

$$\log n_c = \log\frac{n_0}{2\sqrt{\pi Dt}} + rt - \frac{(x-\xi)^2}{4Dt}.$$

If we consider be the density to this critical level

$$\begin{aligned}
\log n_c &= \log \frac{n_0}{2\sqrt{\pi D t}} + rt - \frac{x^2}{4Dt} \\
\Rightarrow \quad \log \frac{n_c 2\sqrt{\pi D t}}{n_0} &= rt - \frac{x^2}{4Dt} \\
\Rightarrow \quad \log \frac{n_c 2\sqrt{\pi D t}}{n_0} &= t\left(r - \frac{x^2}{t^2} \times \frac{1}{4D}\right) \\
\Rightarrow \quad \frac{x^2}{t^2} - 4Dr + \frac{4D}{t} \log\left(\frac{n_c 2\sqrt{\pi D t}}{n_0}\right) &= 0 \\
\Rightarrow \quad \frac{x}{t} = \pm\sqrt{4Dr - \frac{4D}{t}\log\left(\frac{n_c 2\sqrt{\pi D t}}{n_0}\right)}. &\qquad (9.1.42)
\end{aligned}$$

Taking $\lim_{t\to\infty}$,

$$\frac{x}{t} \to \pm 2\sqrt{Dr}. \qquad (9.1.43)$$

The left hand side of (9.1.42) and (9.1.43) can be interpreted as the average velocity of expansion.

This average velocity of expansion tends towards a constant determined by the intrinsic growth rate of the population and diffusive coefficient.

UNIT-11

Autonomous and Non-Autonomous System: Orbit of a map, fixed point, equilibrium point, periodic point, circular map, configuration space and phase space.

11.1 Introduction:

Most of the problems in physical biological and social science involved non-linear differential equation. In the present chapter, we introduce some central ideas and method of the subject and find that it gives some interesting new phenomenon that does not appearing linear theory. The qualitative study of differential equation is concerned with how to deduce improvement characteristic of the solutions of differential equations without solving them. Here we introduce a geometric device, the phase space which is used extensively for obtaining directly from the differential equation such as properties as equilibrium periodicity unlimited growth, stability and so on.

We give some examples of non-linear differential equation arising in practice. The equation of motion for a simple problem is

$$\ddot{x} + \frac{g}{l} \text{Sin } x = 0 \text{ ……………..} (11.1)$$

Where ‘x’ is the inclination of the string of length ‘l’ of downward vertical. If there is a damping force proportional to the velocity of the bob of mass m, then is a equation becomes-

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{g}{l} \text{Sin } x = 0 \text{ ……………..} (11.2)$$

for small angular deviation, Sin x ≈ x in **(11.1)** and **(11.2)** but there involves a gross error.

Another non-linear differential equation is involved in the theory of vacuum tube and is called Vanderpol equation given by

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0 \text{ ……………..} (11.3)$$

where the non-linearity occurs in the second term and μ is the paramiter.

And n-th order non-linear differential equation is of the form-

$$x^{(n)} = \frac{d^n x}{dt^n} = f(t, x, \dot{x}, \ldots, x^{(n-1)}) \text{………………} (11.4)$$

where $\dot{x} = = \frac{dx}{dt}$, $\ddot{x} = = \frac{d^2x}{dt^2}$,…………..etc

Let us write x_1 = x, $x_2 = \dot{x} = \dot{x_1}$,…,$x_4 = \dot{x_3}$, …,$x_n = \dot{x_{n-1}}$.

Thus, $\dot{x_{n-1}}$ = f($x_1, x_2, \dots, x_n$),

f:$R^{n+1} \to R$ .... (11.5)

The system (1.5) is the special case of the following system of n-dimensional equations of first order differential equation

$\dot{x_1}= f_1(t, x_1, x_2, \dots, x_n)$, $\dot{x_2}= f_2(t, x_1, x_2, \dots, x_n)$,…, $\dot{x_n}= f_n(t, x_1, x_2, \dots, x_n)$

or in complete form $\dot{x_i}= f_i(t, x_1, x_2, \dots, x_n)$ for i=1,2,…,n. which in vector form can be written as

$\dot{\vec{x}} = \vec{f}(\vec{x}, t)$ ....(11.6)

with $\vec{x} = (x_1, x_2, \dots, x_n)$ and $\vec{f} = (f_1, f_2, \dots, f_n)$

In study vector function in $R^n = \sum_{i=1}^{n} |f_i|$

For the nxn matrix A with elements a_{ij}, We shall use the norm

$||A|| = \sum_{i=1}^{n} |a_{ij}|$

11.2 Existence and uniqueness:

Regarding the existence and uniqueness and continuity of a solution of differential equation (1.6) the vector function $\vec{f}(\vec{x}, t)$ has to satisfy the following condition, called Lipschitz's condition.

Lipschitz's condition:

Consider the function $\vec{f}(\vec{x}, t)$ with f:$R^{n+1} \to R|t - t_0| \le a$, $\vec{x} \in$ D $\subset R^n$. Then $\vec{f}(\vec{x}, t)$ is said to satisfy the Lipschitz's condition with respect to x if in $[t_0 - a, t_0 + a]$ x D, we have

$$\left\|\vec{f}(\overrightarrow{x_1}, t) - \vec{f}(\overrightarrow{x_2}, t)\right\| \le \alpha ||\overrightarrow{x_1} - \overrightarrow{x_2}||$$

with $\overrightarrow{x_1}$, $\overrightarrow{x_2} \in$ D and α is constant, called the Lipschitz's constant

Instead of saying that $\vec{f}(\vec{x}, t)$ satisfies the Lipschitz's condition, we sometimes say that $\vec{f}(\vec{x}, t)$ is called Lipschitz's condition in $\vec{x}$.

We now state the following theorem without proof.

Theorem 11.1(Cauchy-Lischitz Theorem):

Consider the initial value problem

$\dot{\vec{x}} = \vec{f}(\vec{x}, t)$, $\vec{x}(t_0) = \overrightarrow{x_0}$

with $\vec{x} \in D \subset R^n$, $|t - t_0| \leq a$.

D = { $\vec{x}/\ \|\overrightarrow{x_1} - \overrightarrow{x_2}\| \leq d\}$, a and d are position constants. Suppose that the vector function $\vec{f}(\vec{x}, t)$ satisfies the following conditions

(i) $\vec{f}(\vec{x}, t)$ is continuous in G = $[t_0 - a, t_0 + a]$ x D.
(ii) $\vec{f}(\vec{x}, t)$ is Lipschitz's continuous in $\vec{x}$.

Then the initial value problem has one and only solution for

$|t - t_0| \leq$ inf (a, d/m), M = $\sup_{G} \|\vec{f}\|$

Note that the theorem (1.1) of the existence of the solution neighbourhood of t = t_0. As the n-th order non-linear differential equation is equivalent to the system of n first order differential equation

$$x^n = \frac{d^n x}{dt^n} = f(t, x, x^1, \dots, x^{n-1})$$

is determine uniquely by the values prescribed for x and its 1^{st} (n-1) derivatives at t = t_0, provided that the function satisfies the Lipschitz's condition.

11.3: Gronowall's Inequality:

Theorem 11.2(Gronowall):

Assume that for $t_0 \leq t \leq t_0 + a$, where a is positive constant, we have the estimate

$\varphi(t) \leq \delta_1 \int_{t_0}^{t} \Psi(s)\varphi(s)ds + \delta_2$ .... (11.7)

In which for $t_0 \leq t \leq t_0 + a$, $\varphi(t)$ and $\Psi(t)$ are continuous function, $\varphi(t) \geq 0$ and δ_1, δ_2 position constant. Then we have for $t_0 \leq t \leq t_0 + a$

$\varphi(t) \leq \delta_2 e^{\delta_1 \int_{t_0}^{t} \Psi(s)ds}$ .... (11.8)

Proof:

From (11.7) we get

$$\frac{\varphi\,(t)}{\delta_1 \int_{t_0}^{t} \Psi(s)\varphi(s)ds + \delta_2} \leq 1$$

Multiplying both sides by $\delta_1\ \Psi(t)$ and integrating we have

$$\int_{t_0}^{t} \frac{\delta_1 \Psi(s)\varphi(s)ds}{\delta_1 \int_{t_0}^{t} \Psi(s)\varphi(s)ds + \delta_2} \leq \delta_1 \int_{t_0}^{t} \Psi(s)ds$$

$\Rightarrow \varphi\,(t)\ \leq\ \delta_2 e^{\delta_1 \int_{t_0}^{t} \Psi(s)ds}$

11.4 Autonomous and Non-autonomous system:

Consider the system of n-differential equations of first order;

$\dot{x}_i = f_i(t, x_1, x_2, \ldots, x_n)$ for i=1,2,…,n. which in vector form can be written as

$\dot{\vec{x}} = \vec{f}(\vec{x}, t)$ .... (11.9)

If the vector function $\vec{f}$ occurring in (1.9) depends on $\vec{x}$ only and not on time t, the system given by

$\dot{\vec{x}} = \vec{f}(\vec{x})$ ....(11.10)

is said to be autonomous. On the other hand, if the time 't' appears explicitly in (1.9), the system is said to be non-autonomous.

11.5Phase-Space, Orbits in Autonomous System:

We start with a simple but important property of autonomous equation

$\dot{\vec{x}} = \vec{f}(\vec{x})$ .... (11.11)

Translation Property:

Suppose that $\vec{x}$(t) is a solution of equation (11.11) in the domain D∈ R^n. Then $\vec{x}$(t- t_0) with t_0 is a constant is a also a solution.

Proof:

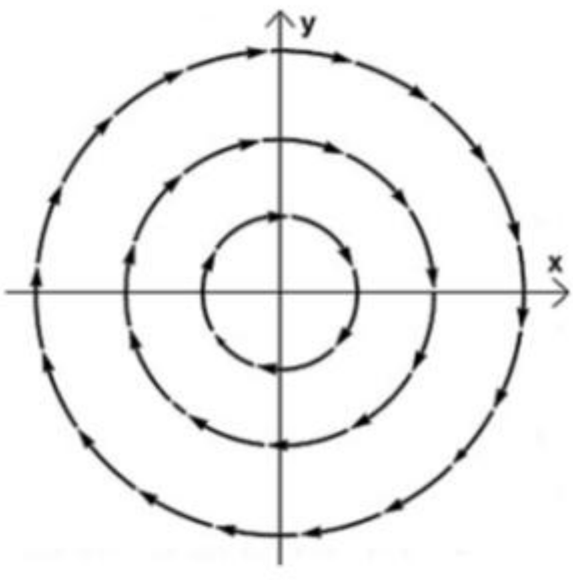
Transform t→T with T = t- t_0 apart from replacing t by T, equation (11.11) does not change as t does not occur explicitly in the right hand side. Since $\vec{x}$(t) is a solution of (11.11), so $\vec{x}$(T) is a solution of the transform equation.

Notes:

(i) It follows that if initial values problem $\dot{\vec{x}}= \vec{f}(\vec{x})$, $\vec{x}(0) = \vec{x}_0$ has the solution of $\vec{x}$(t) then the initial value problem $\dot{\vec{x}}= \vec{f}(\vec{x})$, $\vec{x}(t_0) = \vec{x}_0$has the solution $\vec{x}$(t- t_0).

(ii) It is to be noted that since occurs explicitly in the right hand side of the non-autonomous system $\dot{\vec{x}}= \vec{f}(\vec{x}$,t), so translation property does not hold for such a system.

Now consider the equation (11.1) with $\vec{x} \in D \subset R$, D is called phase-space and for autonomous equation it makes sense to study this space separately. Consider the harmonic equation $\ddot{x} + x = 0$ which is autonomous. To obtain the corresponding vector equation we put x = x_1, $\dot{x} = x_2$ to obtain $\vec{x}_1 = x_2$, $\vec{x}_2 = -x_1$. This solutions of the scalar equation are linear combinations of constant. It is easy to space G = $R\ x\ R^2$. This solutions can be projected on the $\dot{x}$, $\dot{x}$-phase which we call the phase-plane.



As time does not occur explicitly in equation (11.11), we carry out this projection for the solutions of this general autonomous equation. The space in which we describe the behaviour of the variable $x_1, x_2, \dots, x_n$ parameterized by t, is called the phase space. A point in phase space with co-ordinate $x_1(t)$, $x_2(t)$,…,$x_n(t)$ for contain t, is called a phase point. In general, for increasing t, a phase point shall more through phase-space.

Consider the equation (11.11) which written out in components becomes

$\dot{x}_i$= $f_i(x_1, x_2, \dots, x_n)$ for i=1,2,…,n.

We shall use one of the components, say f_1, as a new independent variable; this required that $f_1(x_1, x_2, \dots, x_n) \neq 0$. With the chain rulewe obtain (n-1) equations

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2,\dots,x_n)}{f_1(x_1,x_2,\dots,x_n)}, \dots, \frac{dx_n}{dx_1} = \frac{f_n(x_1,x_2,\dots,x_n)}{f_1(x_1,x_2,\dots,x_n)} \text{ .... (11.12)}$$

Solution of system (11.12) in phase-space are called orbits. In the existence and uniqueness theorem (11.1) applies to the autonomous equation (11.11), it also applies to the system (11.12), describing the behaviour of the orbits in phase space will not intersect. For such a case the point $(x_1, x_2, \dots, x_n)$ on the orbit is called an ordinary point.

In the above we have excluded the singularities of the right hand side of system (11.12) corresponding with the zeros of$f_1(x_1, x_2, \dots, x_n)$. If $f_2(x_1, x_2, \dots, x_n) \neq 0$, we interchanged the results of f_1 and f_2. If the zero of f_1 and f_2 consider, we can take as x_3 as independent variable, etc.

Real problem with this construction arise in points $\vec{a} = a_1, a_2, \dots, a_n$ such that $f_1(\vec{a}) = 0, f_2(\vec{a}) = 0, \dots, f_n(\vec{a}) = 0$.

The point $\vec{a} \in R$ is the zero of the vectors function $\vec{f}(\vec{x})$ and we call it a critical point or singular point or an equilibrium point.

Example 11.1: Consider the harmonic oscillator $\ddot{x} + x = 0$. The equivalent vector equation is with x = x_1, $\dot{x} = x_2, \dot{x_1} = x_2, \dot{x_2} = -x_1$

The phase space is two dimensional and (0,0) is the critical point. The orbits are described by the equation

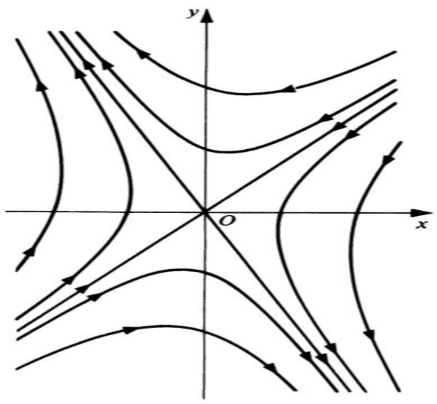
$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

or $x_1\, dx_1 + x_2\, dx_2 = 0$.

or $x_1^2 + x_2^2$ =constant = C.

Thus the orbits are concentric circles in the phase-space.

Example 11.2: The equation $\ddot{x} - x = 0$ gives the orbits the phase-plane as $\frac{dx_2}{dx_1} = \frac{x_1}{x_2}$. Integration gives the family of hyperbolas $x_1^2 - x_2^2$ =constant = C. The critical point is (0,0)



The arrows indicate the direction of motion of the phase points with time, the motion of phase points along the corresponding is called the phase-flow.

1.6 Critical point and linearization:

Consider the equation (11.11)

$\dot{\vec{x}} = \vec{f}(\vec{x}), \vec{x} \in D \subset R^n$

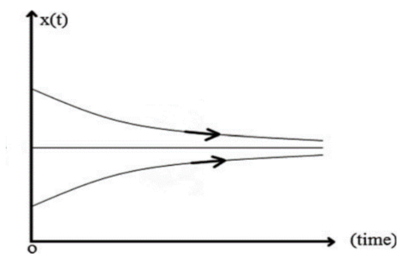
and we assume that the vector function $\vec{f}(\vec{x})$ has a zero at $\vec{x} = \vec{a}$ with $\vec{f}(\vec{a}) = 0$ is a critical point of the equation $\dot{\vec{x}} = \vec{f}(\vec{x})$. A critical point of the equation in phase space can be considered as an orbit, degenerated into a point

Note that a critical point corresponds with an equilibrium solution (or stationary solution) of the equation for all time.

It follows form the existence and uniqueness theorem (1.1) that an equilibrium solution never be reached into a finite time (if an equilibrium solution would be reached in a finite time, two solutions would be intersect).

Example 11.3:

Consider the equation $\dot{x} = -x, t \geq 0$.



$x = 0$ is a critical point, x(t) = 0, $t \geq 0$ is equilibrium solution.

Note that for solutions, starting in $x_0 \neq 0$ at t = 0 given by x(t) = $x_0 e^{-t}$.

We have, $\lim_{t\to\infty}$ x(t) $= 0$

Example 11.4:

Consider the equation $\dot{x} = -x^2 t \geq 0$.

Now x = 0 is a critical point and x (t) = 0, t ≥0 is an equilibrium solution. The solution starting in $x_0 \neq 0$ $at\ t = 0$ and x (t) = $\frac{1}{x_0^{-1}+t}$, $x_0 \neq 0$.

If $x_0 < 0\ and\ x_0 > 0$, the solutions shows qualitative and quantities difference behaviour for the two cases. If $x_0 < 0$, the solutions become unbounded in a finite time.

In example (11.3), the solutions tend in the limit for t → ∞ towards the equilibrium solution, the orbits in the one dimensional phase space tends to critical point. If we consider the equation $\dot{x} = x$, $t \geq 0$, $x_0 \neq 0$ at t = 0, then the solution will be x(t) = $x_0 e^t$ so that x(t) → 0 as t → $-\infty$ i.e., the orbits tend away from the critical point. We call these phenomenon attractions.

A critical point $\vec{x} = \vec{a}$ of the equation $\dot{\vec{x}} = \vec{f}(\vec{x})$ in R^n is called positive attraction if there exists neighbourhood $\Omega_a \subset R^n$ of $\vec{x} = \vec{a}$ such that $\vec{x}(t_0) \in \Omega_a$ implies $\lim_{t\to\infty} \vec{x}(t) = \vec{a}$.

If a critical point has this property for t → $-\infty$ then $\vec{x} = \vec{a}$ is called negative attractor.

In analysing critical points and equilibrium solutions we start by linearising the equation in a neighbourhood of the critical point. We assume that $\vec{f}(\vec{x})$ has a Taylor series expansion of the first degree plus higher order rest terms. So, in the case of $\dot{\vec{x}} = \vec{f}(\vec{x})$, we write in the nbd of critical point $\vec{x} = \vec{a}$.

$\dot{\vec{x}} = \vec{f}(\vec{x}) = \vec{f}(\vec{a} + \vec{x} - \vec{a}) = \frac{\partial \vec{f}(\vec{a})}{\partial \vec{x}} (\vec{x} - \vec{a})$+ higher order term.

We shall study linear equation with constant coefficient

$$\dot{\vec{x}} = \frac{\partial \vec{f}(\vec{a})}{\partial \vec{x}} (\vec{x} - \vec{a})$$

To simplify the notation, the point notation, the point $\vec{a}$ is often shifted to the origin of phase space. Thus by putting $\vec{\xi} = (\vec{x} - \vec{a})$ we have

$$\dot{\vec{\xi}} = \frac{\partial \vec{f}(\vec{a})}{\partial \vec{\xi}} \vec{\xi}$$

Let, $\frac{\partial \vec{f}(\vec{a})}{\partial \vec{\xi}} = A$, a nxn matrix with constant coefficient. So, the linearized system which we shall study in the nbd of $\vec{x} = \vec{a}$ is of the form $\dot{\vec{\xi}} = A\vec{\xi}$.

Example 11.5:

Consider the equation

$\ddot{x} + \sin x = 0$ with $-\pi \leq x \leq \pi, x \in R$.

Let x = x_1, $\dot{x} = x_2$ then the above equation reduces to $\dot{x_1} = x_2, \dot{x_2} = -\sin x_1$.

Critical points are (x_1, x_2) = (0, 0),(-π, 0),(π, 0).

Expansion in of (0, 0) gives

$\dot{x_1} = x_2, \dot{x_2} = -x_1 + higher\ order\ term$ and that in the nbd of $(\pm\pi, 0)$ gives

$$\dot{x_1} = x_2, \dot{x_2} = -x_1 \mp \pi + heigher\ order\ term.$$

Example 11.6:

Consider the system describing the interacting of the two species

$\dot{x} = ax - bxy$

$$\dot{y} = bxy - cy$$

With x, y $\geq$ 0

And a, b, c are constants.

'x' denotes the population density of the prey 'y' the population density of the predator.

In this model the survival of the predator depends completely on the presence of prey; to put mathematically if x(0) = 0, we have $\dot{y} = -cy$so that y(t) = y(0) e^{-et} and

$$\lim_{t\to\infty} \text{y(t)} = 0$$

The equilibrium solution corresponds with the critical points (0,0) and $(\frac{c}{b}, \frac{a}{b})$ is

$\dot{x} = -c(y - \frac{a}{b})$+…..

$\dot{y} = -a(x - \frac{c}{b})$+…..

1.7Periodic Solutions:

Suppose that $\vec{x} = \vec{\varphi}(t)$ is the solution of the equation $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in D \subset R^n$ and suppose that there exists a positive number T such that $\vec{\varphi}(t + T) = \vec{\varphi}(t)$ for all t $\in R$.

Then $\vec{\varphi}(t)$ is called periodic solution of the equation with period T. if $\vec{\varphi}(t)$ has period T, then the solution has also period 2T, 3T, ……..Suppose T is the smallest period, then we called $\vec{\varphi}(t)$, T-periodic.

Lemma 11.1:

The periodic solution of the autonomous equation $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in D \subset R^n$, t $\in$ R, corresponding with a closed orbit corresponds with a periodic solution.

Proof:

Consider the phase space corresponding with $\dot{\vec{x}} = \vec{f}(\vec{x})$. For a periodic solution we have that often time T, $\vec{x} = \vec{\varphi}(t)$, assume the same value in R^n. So, a periodic solution products a closed orbit or cycle in phase space.

Consider now a closed orbit C in phase space and a point $\vec{x}_0 \in C$. The solution of the equation $\dot{\vec{x}} = \vec{f}(\vec{x})$ given by $\vec{\varphi}(t) = \vec{0}$ stands at t=0 in $\vec{x}_0$ and the traces the orbit C because of uniqueness of solutions, C can not contain a critical point, so $\|\vec{f}(\vec{x})\| \geq a > 0$ for $\vec{x} \in C$.

Therefore, $\|\dot{\vec{x}}\| \geq a > 0$

So that at a certain time t=T, we have returned to $\vec{x}_0$. Now we have to show that $\vec{\varphi}(t+T) = \vec{\varphi}(t)$ for all t$\in R$.

Let t = nT+t_1 with n$\in Z$, 0 <t_1< T.

It follows from the translation property (see article 1.5) $\vec{\varphi}(t)$ is a solution $\vec{\varphi}(t) = \vec{x}_1$ then $\vec{\varphi}(t-nT)$ is a solution with $\vec{\varphi}(t+nT)$=$\vec{x}_1$

So, $\vec{\varphi}(t_1)$ = $\vec{\varphi}(t_1 + nT)$

and as t_1 can have any value between (0, T), we see that $\vec{\varphi}(t)$ is T-periodic.

Example-11.7:

For the equation $\ddot{x} + \sin x = 0,$

The phase plane contains a family of closed orbits corresponding with periodic solutions.

Example-11.8:

For the Vander Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \mu > 0$$

The phase plane contains one closed orbit corresponding with periodic solution.

Note: The definition of periodic solution also applies to solutions of non-autonomous equations $\dot{\vec{x}} = \vec{f}(\vec{x}, \mathrm{t})$. However, closed orbits in such a system do not necessarily corresponds with periodic solution property is not valid any more.

Consider for example the system

$$\dot{x} = 2 + y$$

$$\dot{y} = -2tx$$

where solutions on of the form

x (t) = $\alpha \cos t^2 + \beta \sin t^2$

y(t) = $-\alpha \sin t^2 + \beta \cos t^2$

In the xy-phase plane we have closed orbits, but the solutions are not periodic.

1.8 Orbital Derivative:

Consider the differentiable function $\vec{F}: R^n \to R$ and the vector function $\vec{x}: R \to R^n$. The L_t of the function $\vec{F}$ alone the vector function $\vec{x}$, parameterise by t is

$$L_t\vec{F} = \frac{\partial \vec{F}}{\partial \vec{x}}\dot{\vec{x}}$$

$$= \frac{\partial \vec{F}}{\partial \overrightarrow{x_1}}\dot{x}_1 + \frac{\partial \vec{F}}{\partial \overrightarrow{x_2}}\dot{x}_2 + \ldots + \frac{\partial \vec{F}}{\partial \overrightarrow{x_n}}\dot{x}_n$$

where $x_1, x_2, \ldots, x_n$ are the components of $\vec{x}. L_t$ is called the orbital derivatives.

Now we chose for $\vec{x}$-solution of the differential equations $\dot{\vec{x}}= \vec{f}(\vec{x})$ to compute the orbital derivative.

First Integrals and Integral Manifolds: Consider the equation $\dot{\vec{x}}= \vec{f}(\vec{x})$, $\vec{x} \in D \subset R^n$, the function $\vec{F}(\vec{x})$ is called the first integral if D holds $L_t\vec{F} = \overrightarrow{0}$.

It follows from the definition that the first integral $\vec{F}(\vec{x})$ is constant alone a solution. First integral are same times called constant of motion.

Taking $\vec{F}(\vec{x})$ = constant, we are considering the level sets contains orbits of the equation such a level set defined by $\vec{F}(\vec{x})$ = Constant consists of family of orbits called an integral manifold.

As for an example consider the equation $\ddot{x} + x = 0$.

The first integral is

$\frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2 = E,\ \ E \geq 0$,a constant determined by the initial conditions. In phase space, this relation corresponds for E>0 with manifold, a circle around the origin.

Definition:

Invariant Set: Consider the equation $\dot{\vec{x}}= \vec{f}(\vec{x})$, $\vec{x} \in D \subset R^n$. The set M$\subset D$ is invariant if the solution $\vec{x}(t)$ with $\vec{x}(0)$ $\in$ M for $-\infty < t < \infty$. If this property is valid only for t $\geq 0 (\leq 0)$ then M is called a positive (a negative) invariant set.

Critical points and in general solutions which exists for all time are examples of invariant sets.

Non Degenerate Critical Point of $\vec{F}(\vec{x})$: Consider $\vec{F}: R^n \to R$ which is supposed to be C^∞, for $\vec{x} = \vec{a}$ we have

$\left(\frac{\partial \vec{F}}{\partial \vec{x}}\right)_{\vec{x}\,=\,\vec{a}}$ = $\vec{0}$ and $\vec{x}$ = $\vec{a}$ is a critical point.

The point $\vec{x}$ = $\vec{a}$ is called non-degenerate critical point of the function $\vec{F}(\vec{x})$ if we have the determinant

$$\left(\frac{\partial^2 \vec{F}}{\partial \overrightarrow{x^2}}\right)_{\vec{x}\,=\,\vec{a}} \neq 0.$$

for example the origin is non-degenerate critical point of the functions $x_1^2 + x_2^2, x_1^2 - 2x_2^2$and the origin is a degenerate critical point of the functions $x_1^2 x_2^2$, $x_1^2 + x_2^3$.

Definition:

Morse Function: If $\vec{x}$ = $\vec{a}$ is non-degenerate critical point of the C^∞ function in the neighbourhood of $\vec{x}$ = $\vec{a}$.

It is to be noted that the behaviour of a Morse function in a nbd of critical point $\vec{x}$ = $\vec{a}$ is determined by the quadratic part of the Taylor expression of the function. Suppose that $\vec{x}$ = $\vec{0}$ is a non-degenerate critical point of the Morse function $\vec{F}(\vec{x})$ with

$\vec{F}(\vec{x})$ = $\overrightarrow{F_0} - c_1x_1^2 - c_2x_2^2 - \cdots - c_kx_k^2 + c_{k+1}x_{k+1}^2 + \cdots + c_nx_n^2$+ higher order terms.

with the coefficients $c_1, c_2, \ldots, c_n$; k is called the index of the critical points. There exists a transformation $\vec{x} \to \vec{y}$ in a nbd of the critical point such that $\vec{F}(\vec{x}) \to \vec{G}(\vec{y})$ where $\vec{G}(\vec{y})$ is also Morse function with critical point $\vec{y} = 0$, the same index k and apart from $\vec{G}(\vec{0})$ only quadratic terms.

Lemma:

Consider the function $\vec{F}: R^n \to R$ with non-degenerate critical point $\vec{x}$ = $\vec{0}$, index k. In a nbd of $\vec{x}$ = $\vec{0}$ there exists a diffeomorphism (transformation which is one-to-one, unique C' and which the inverse exists and is also C') which transforms $\vec{F}(\vec{x})$to the form

$$\vec{G}(\vec{y}) = \vec{G}(\vec{0}) - y_1^2 - y_2^2 - \cdots y_k^2 + y_{k+1}^2 + \cdots y_n^2$$

11.9Evaluation of Volume Element:

Consider the equation $\dot{\vec{x}}$= $\vec{f}(\vec{x})$ in R^n and a domain D(0) in R^n which is suppose to have volume v(0). The flow defines a mapping $\vec{g}$ of D(0) into R^n,

$\vec{g}: R^n \to R^n, D^t = g^t D(0)$

From the volume v(t) of the domain D(t) we have

$\frac{dv}{dt}]_{t=0}$ = $\int \vec{\nabla}.\vec{f} d\vec{x}$ ($\vec{\nabla}.\vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n}$)

11.10 Characterization of Critical Points:

In section (11.6) we have seen that linearization in a nbd of a critical point of an autonomous system $\dot{\vec{x}} = \vec{f}(\vec{x})$ leads to the equations

$\dot{\vec{\xi}} = A\vec{\xi}$ ………………(11.13)

where A is a constant nxn matrix; in this formulation the critical point has been translated to the origin. We assume that $|A| \neq 0$ and critical point is non-degenerate.

The eigen values of A are obtained from the characteristic equations

$|A - \lambda I| = 0$……………….(11.14)

Let the eigen values are $\lambda_1, \lambda_2, \dots, \lambda_n$. In the eigen values are different then there exists a real, non-singular matrix T such that $T^{-1}AT$ is a diagonal matrix whose diagonal elements are the eigen values. If there are some equal eigen values, the linear transformation $\vec{\xi} = T\vec{z}$ leads to

$T\dot{\vec{z}} = AT\vec{z}$

i.e $\dot{\vec{z}} = T^{-1}AT\vec{z}$ ……………………….(11.15)

which can be integrated from which $\vec{\xi} = T\vec{z}$ follows.

Two Dimensional Linear System:

We shall give the location of the eigen values by the diagram which consists of the complex plane (real axis horizontally, imaginary axis vertically), where the eigen value are indicated by dots.

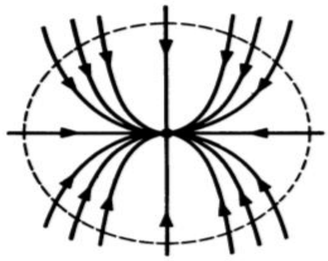
Here the dimension is two and eigen values $\lambda_1 \, and \, \lambda_2$ are both real or complex conjugate. If $\lambda_1 \neq \lambda_2$ (real or conjugate). Then $T^{-1}AT$ is of the form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

We find for Z(t) the general solution $\vec{Z}(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$ …………..(11.16)

Where $c_1 \, and \, c_2$ are arbitrary constant. The behaviour of the solutions represented by (11.16) is for the kind of choices of $\lambda_1 \, and \, \lambda_2$ very different. We have the following case

(a) Two Node (λ_1 *and* λ_2 Real and Same Sign):

The eigen value are real and have the same sign. If $\lambda_1 \neq \lambda_2$ we have with $\vec{Z}(t)=(z_1, z_2)$ the real solution $z_1(t) = c_1 e^{\lambda_1 t}$ and $z_2(t) = c_2 e^{\lambda_2 t}$. Elimination of t produces $|z_1|$ = c $|z_1|^{\frac{\lambda_1}{\lambda_2}}$ with c being constant. So in the phase-plane we find orbits which are related to parabola.

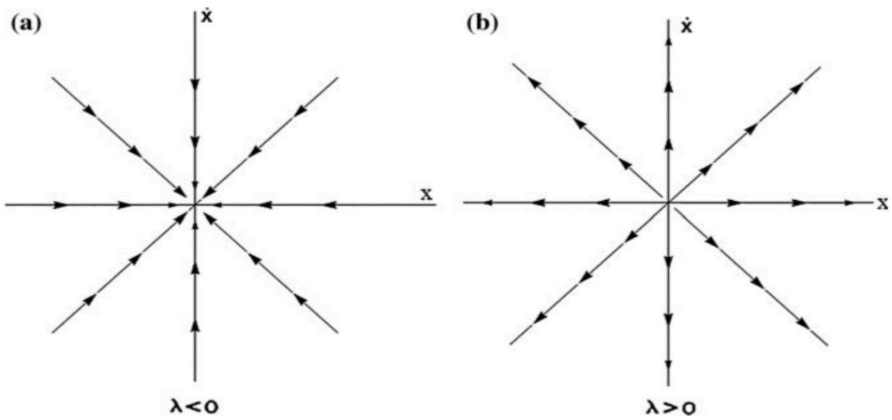


Nodes

We call this critical point is a node. If $\lambda_1, \lambda_2 < 0$ then (0,0) is a positive attractor and a stable node; if $\lambda_1, \lambda_2 > 0$, then (0,0) is a negative attractor and a unstable node. If $\lambda_1 = \lambda_2 = \boldsymbol{\lambda}$ (say) the normal form is $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ so that $z_1(t) = (c_1 + c_2)e^{\lambda t}$ and $z_2(t) = c_2 e^{\lambda t}$.

Here the critical point is called inflected node, a positive attractor if $\boldsymbol{\lambda}<0$ (stable) and a negative attractor (unstable) if $\boldsymbol{\lambda}>0$.

If particular, if one root, say λ_2 is zero then the solution describes a family of straight lines through the origin, positive attractor if $\lambda_1 < 0$(stable) and negative attractor if $\lambda_1 > 0$(unstable). The critical point is called a proper node or star.



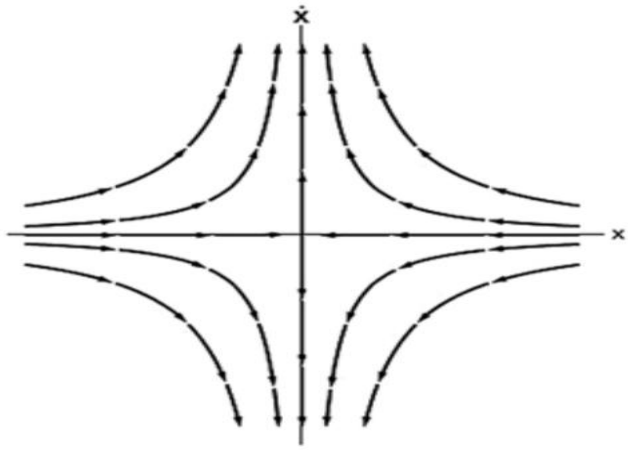
(b) The Saddle Point(λ_1 *and* λ_2 Real and Opposite Sign):

The solutions in this case are again of the form given by the equation(11.16). In the phase-plane the orbits are given by

$$z_1(t) = c_1 e^{\lambda_1 t}$$

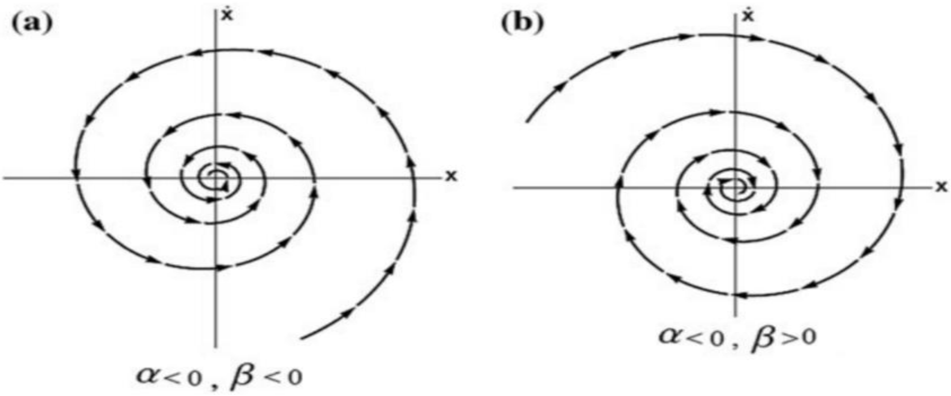
$|z_1| = \text{c}\ |z_1|^{-\left|\frac{\lambda_1}{\lambda_2}\right|}$ with constant.

The behaviour of the orbits is hyperbolic, the critical point (0, 0) is not an attractor. We call this a saddle point (unstable). It should be noted that the co-ordinate axes correspond with five different solutions. The critical point (0, 0) and four half axes.



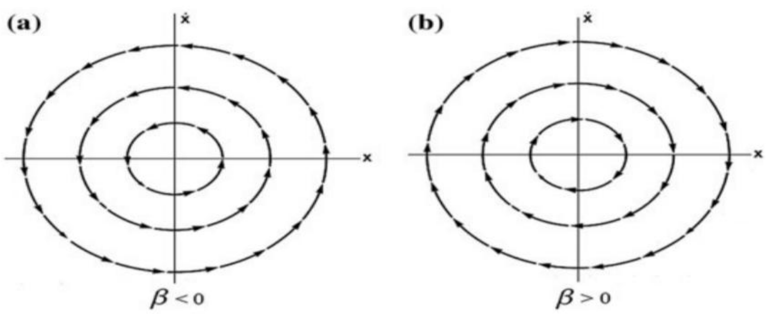
(c) The Spiral or Focus (λ_1 *and* λ_2 Complex Conjugate With Non-Zero Real Part):

Let λ_1, λ_2 = u + i v with u, v $\neq 0$. The solutions are of the form $e^{(u+iv)t}$. Linear Combination of the complex solutions of the form $e^{\mu t}\cos\omega t$ and $e^{\mu t}\sin\omega t$. The orbits are spiralling in or out with respect to (0, 0) and we call the critical point a spiral or focus. If $\mu < 0$, the critical point is positive attractor (stable), if $\mu > 0$, the critical point is a negative attractor (unstable).



(d) The Centre (**λ_1 *and* λ_2 are Purely Imaginary**):

Let $\lambda_1, \lambda_2 = \pm i\omega$ with $\omega \neq 0$ being real. Then (0, 0) is called a centre (stable). The solution can be written as combination of $\cos\omega t$ and $\sin\omega t$. The orbits in the phase-plane are circle. It is clear that (0, 0) is not an attractor.



Critical Points of Non-Linear Equations:

Until now we have analysed critical points of autonomous equations $\dot{\vec{x}} = \vec{f}(\vec{x})$ by linear analysis. We assume that the critical point has been translated to $\vec{x} = \vec{0}$ and that we can write the equation of the form

$\dot{\vec{x}} = A\vec{x} + \vec{h}(\overrightarrow{x})$ …………………..(11.17)

with a non-singular nxn matrix and that

$\lim_{\|\vec{x}\|\to 0} \frac{\|\vec{h}(\overrightarrow{x})\|}{\|\vec{x}\|} = 0$

The nature of the singularity of the critical point of the non-linear system (11.17) is given by the following theorem of Poincare which we state without proof.

Theorem 1.3(Poincare Theorem):

If the critical point $\vec{x} = \vec{0}$ of the linear system $\dot{\vec{x}} = A\vec{x}$ be a node, saddle point or a spiral, then the critical point of non-linear system (11.17) is of same type. On the other hand, if the linear approximation has an inflected node or a proper node at $\vec{x} = \vec{0}$, then the non-linear system can have either a node or a spiral, and if the linear approximation has a centre at $\vec{x} = \vec{0}$, the non-linear system can have either a centre or a spiral.

Example 11.9: Locate the critical point of the non-linear system

$$\dot{x} = -6y + 2xy - 8$$

$$\dot{y} = y^2 - x^2$$

and classify them according to their linear approximation.

Solution: For the critical point we have

$$-6y + 2xy - 8 = 0$$

$$y^2 - x^2 = 0$$

i.e., y = $\pm x$.

When y = x ⇒ $-6y + 2x^2 - 8 = 0$ ⇒ x = 4, 1 and y = 4, -1.

The critical point are (4, 1) and (4, -1).

When y = $-$x ⇒ $6x - 2x^2 - 8 = 0$ ⇒ x = $\frac{3 \pm i\sqrt{7}}{2}$

Which are complex and therefore it is omitted.

For the critical point (4, 4), we put

$$x = \xi + 4 \text{ and } y = \eta + 4$$

Therefore the given system gives

$\dot{\xi} = -6(\eta + 4)^2 + 2(\xi + 4)(\eta + 4) - 8$

and $\dot{\eta} = (\eta + 4)^2 - (\xi + 4)^2$

i.e., $\dot{\xi} = 8\xi + 2\eta + 2\xi\eta$

$$\dot{\eta} = -8\xi + 8\eta - \xi^2 + \eta^2$$

Linear approximation is

. $\dot{\xi} = 8\xi + 2\eta$

$$\dot{\eta} = -8\xi + 8\eta$$

$$\therefore A = \begin{pmatrix} 8 & 2 \\ -8 & 8 \end{pmatrix}$$

The characteristic equation is $\begin{vmatrix} 8-\lambda & 2 \\ -8 & 8-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - 16\lambda + 80 = 0$$

$$\Rightarrow \lambda = 8 \pm 4i$$

Since the eigen values are complex conjugate with non-zero real part, so the critical point is an unstable spiral and negative attractor. Thus the Poincare Theorem, the critical point of the given non-linear system is of same type.

For the critical point $(-1, -1)$, we put

$$x = \xi - 1 \text{ and } y = \eta - 1$$

The given system gives

$$\dot{\xi} = -6(\eta - 1) + 2(\xi - 1)(\eta - 1) - 8$$

$$\dot{\eta} = (\eta - 1)^2 - (\xi - 1)^2$$

i.e., $\dot{\xi} = -2\xi - 8\eta + 2\xi\eta$

$$\dot{\eta} = 2\xi - 2\eta - \xi^2 + \eta^2$$

Linear approximation is

$$\dot{\xi} = -2\xi - 8\eta$$

$$\dot{\eta} = 2\xi - 2\eta$$

The characteristic equation is

$$\begin{vmatrix} -2-\lambda & -8 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -2 \pm 4i$$

Since the given eigen values are complex conjugate with non-zero negative real part so that the critical point is stable spiral. Thus by Poincare theorem, the critical point (-1, -1) of the given non-linear is of the same type.

Exercises:

Exercise-1:

Locate the critical points and find their nature for the following non-linear system:

a) $\dot{x} = -2x - y + 2, \dot{y} = xy$; $(1, 0)$ saddle point, $(0, 2)$ saddle points.
b) $\dot{x} = 4 - 4x^2 - y^2, \dot{y} = xy$; $(0, \pm 2)$ centre, $(\pm 1, 0)$ saddle points.
c) $\dot{x} = siny, \dot{y} = x + x^2$; $(0, n\pi)$, saddle points if n is even and centres if n is odd.
d) $\dot{x} = x^2 - y, \dot{y} = x - y$; $(1, 1)$ saddle points and $(0, 0)$ stable spiral.

Exercise-2: For the equation of motion of the damped pendulum $\ddot{x} + \frac{c}{m}\dot{x} + \frac{g}{a}\sin x = 0 (m > 0, c > 0)$ investigate the nature of critical point.

Solution: Let $\dot{x} = y$, so that $\dot{y} = -\frac{c}{m}y - \frac{g}{a}sinx$

$$= -\frac{g}{a}x - \frac{c}{m}y + \frac{g}{a}(x - sinx)$$

Now $\frac{|x - sinx|}{\sqrt{x^2+y^2}} \leq \frac{|x - sinx|}{|x|} = \left|\mathbf{1} - \frac{\sin x}{a}\right| \to 0$ as $(x, y) \to (0, 0)$

Linear approximation is

$$\dot{x} = y, \dot{y} = -\frac{g}{a}x - \frac{c}{m}y.$$

$(0, 0)$ is the critical point.

The characteristic matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -g/a & -c/m \end{pmatrix}$$

The characteristic equation is

$$\begin{vmatrix} \lambda & 1 \\ -\frac{g}{a} & -\frac{c}{m} - \lambda \end{vmatrix} = 0.$$

$$\Rightarrow \lambda = \frac{\sqrt{ac} \pm \sqrt{ac^2 - 4gm^2}}{2m\sqrt{a}}$$

Critical point is a stable node (the attractor), if $ac^2 > 4gm^2$, stable spiral(the attractor) if $ac^2 < 4gm^2$and inflected node(the attractor) if $ac^2 = 4gm^2$.

Exercise-3: Investigate the nature of the critical point for the Van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ for the cases $\mu > 0$ and $\mu < 0$.

Hint: 0 <2< μ; node with $-ve$ attractor.

$-2< \mu < 0$; node with $+ve$ attractor.

$0< \mu < 2$; spiral with $-ve$ attractor.

$\mu < -2 < 0$; spiral with $+ve$ attractor.

Exercise-4: Determine the critical point and there nature for the system $\ddot{x} + \alpha sinx = 0$, where α is constant.

Hint: Critical points are $(n\pi, 0)$.

For $\alpha > 0$, saddle point if n is even and centres and spirals if n is odd. For $\alpha < 0$, centres and spiral if n is even and saddle points if n is odd.

UNIT-12

Non-linear Conservative System: Nonlinear oscillators-conservative system. Hamiltonian system. Various types of oscillators in nonlinear system viz. simple pendulum, and rotating pendulum.

12.1 Conservative Systems:

Let us consider a system with one degree of freedom and let x be a generalised co-ordinate (e. g. position, angle etc.). Let T and V be the K.E and potential energy function and assume that they have the form

$$T = \frac{1}{2} m(x)\dot{x}^2, \; v = v(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12.1)$$

where m(x)(>0) is another function of x. If the system is conservative, the total energy ξ is constant during motion.

i.e., $\frac{1}{2} m(x)\dot{x}^2 + v(x) = \xi$, a constant......................(12.2)

which gives the phase paths.

The type of equation which leads to (12.2) can be obtain by taking the time derivative of (12.2)as follows

$$m(x)\ddot{x} + \frac{1}{2} m'(x)\dot{x}^2 + v'(x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12.3)$$

which can be simplified by introducing a new variable u in a place of x by

$$u = \int \sqrt{m(x)}\, dx$$

$$\therefore \dot{u} = \sqrt{m(x)}\dot{x}$$

and $\ddot{u} = \frac{1}{2} \frac{m'(x)}{\sqrt{m(x)}} \dot{x}^2 + \sqrt{m(x)}\dot{x}$

$$= \frac{m(x)\ddot{x} + \frac{1}{2} m'(x)\dot{x}^2}{\sqrt{m(x)}} = -\frac{v'(x)}{\sqrt{m(x)}}. \quad \ldots\ldots\ldots\ldots\ldots..(12.4)$$

Equation (12.3) then becomes $\ddot{u} + f(u) = 0$ and corresponding energy type equation for the phase path is

$$\frac{1}{2} + \dot{u}^2 + \int f(u)\, du = c \text{ ………………..……(12.5)}$$

in which $f(u) = \frac{v^{'}(x)}{\sqrt{m(x)}}$

Example 12.1

Show that the equations of the form $\ddot{x} + g(x)\dot{x}^2 + h(x) = 0$ are affect ably conservative. Find a transformation which puts the equation into the conservative form.

Solution:

Let u = u(x)

$$\therefore \dot{u} = u'\dot{x} \Rightarrow \dot{x} = \frac{\dot{u}}{u^{'}}$$

Also $\ddot{u} = u^{''}\dot{x}^2 + u^{'}\ddot{x} = u^{''}\frac{\dot{u}^2}{u^{'2}} + u^{'}\ddot{x}$

$$\therefore \ddot{x} = \frac{\ddot{u}\, u^{'2} - u^{''}\dot{u}^2}{\dot{u}^3}$$

Hence the given equation is transformed into

$\frac{\ddot{u}\, u^{'2} - u^{''}\dot{u}^2}{\dot{u}^3}$ + g(x)$\frac{\dot{u}^2}{u^{'2}}$ +h(x) = 0.

or, $\frac{\ddot{u}}{u^{'}} + \{g(x) - \frac{u^{''}}{u^{'}}\}\frac{\dot{u}^2}{u^{'2}}$+h(x) = 0.

We choose g(x) = $\frac{u^{''}}{u^{'}}$ and then the above equation transformed into $\ddot{u} + g(x)u^{'} = 0$ i.e., $\ddot{u} + f(u) = 0$.

Since g(x) = $\frac{u^{''}}{u^{'}}$, we have

$$\log u^{'} = \int g(x)dx$$

or $u^{'} = e^{\int g(x)dx}$

$$\therefore u = \int \{e^{\int g(x)dx}\}dx$$

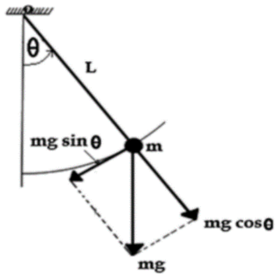
Examples of Non-Linear Conservative System with one Degree of Freedom:

i) **Simple Pendulum:** The motion of simple pendulum of length l, mass m is given by

$$ml\ddot{\theta} + mg\sin\theta = 0.$$

i.e., $\ddot{\theta} + \omega_0^2 \sin\theta = 0 \;\; where\; \omega_0^2 = \frac{g}{l}$ ………………(12.6A)

Here the non-linearity is due to large motion corresponding to large deformation.



ii) **Particle Restrained by Non-Linear Spring:** Consider the motion of the particle of mass m on a horizontal frictionless plane and restrained by the non-linear spring. If x(t) be the position of the mass then the differential equation is

$m\ddot{x} = -f(x)$ …………………………..(12.6B)

where f(x) is the force exerted by the spring on the mass and the force is the non-linear function of the displacement. Here the non-linearity is due to the material behaviour.

iii) **Particle in a Central Force Field:** The equation of motion of the particle moving in a plane under a central force field F(r) is

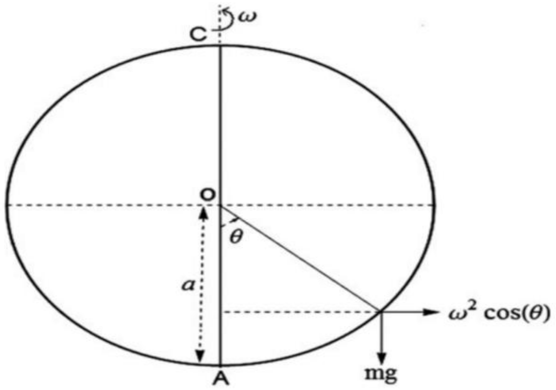
$\frac{d^2u}{d\theta^2} + u = \frac{F(1/u)}{h^2u^2}$, where u = 1/r ……………….(12.6C)

where the field is gravitational and electrical. Here the non-linearity is due to inertia as well as material property.

iv) **Rotating Pendulum(A Particle Rotating in a Circle**): Consider the motion of the particle of mass m moving without friction alone a circle of radius a which rotates with angular velocity Ω about its vertical diameter. The forces acting on the particle are the gravitational force mg, the centrifugal force $m\Omega^2 a\sin\theta$ and reaction force M(say). Taking the moment about the centre O of the circle and equating their some to the rate of change of angular momentum of the particle about O we get

$$ma^2\ddot{\theta} = m\Omega^2 a\sin\theta . a\cos\theta - mga\sin\theta$$

i.e., $\ddot{\theta} + \left(\frac{g}{a}\sin\theta - \Omega^2\sin\theta . \cos\theta\right) = 0$ …………………(12.6D)



Here the non-linearity is due to inertia as well as large deformation. It is seen from the above that all the equations in (12.6) are of the form $\ddot{x} + f(x) = 0$.

12.2 Energy Integral:

Consider the differential equation

$$\ddot{x} + f(x) = 0 \quad \text{……………………….(12.7A)}$$

for non-linear conservative systems with one degree of freedom. The equivalent system is $\dot{x} = y \; and \; \dot{y} = -f(x)$ ………..(12.7B)

and the integral curve is

$$\frac{dy}{dx} = \frac{-f(x)}{y} \quad \text{………………………….(12.8)}$$

which shows that the integral curves have a horizontal tangent at the points x_i, the roots of a equation f(x)=0, provided y $\neq 0$ at this points. For critical points we have simultaneously f(x) = 0, y=0 i.e., the critical points (if exists) lie on the x-axis.

The energy integral is

$$\frac{1}{2}y^2 + v(x) = \xi, a\ constant. \ \ where\ v(x) = \int_0^x f(x)dx \quad \text{……………………….(12.9)}$$

We can consider $\frac{1}{2}y^2 = \frac{1}{2}\dot{x}^2$ as the K.E and v(x) as the potential energy; the constantξ, the total energy shows that the system is conservative. The constant ξ is determined by the initial conditions and is called the energy integral. For a given value of ξ the solution (12.9) represents on the phase plane (xy-plane)a curve which we call integral curve or a level curve or a curve of constant energy.

The behaviours of the level curves are called trajectories. As time passes, the point of the phase plane representing the solution move alone the trajectories. The direction of sence of the motion of the point can be determined by considering the velocity y(=$\dot{x}$). Clearly, x must be increasing function of time if y > 0.

We re-write (12.9) in the form

y = $\pm\sqrt{2\{\xi - v(x)\}}$ …………………(12.10)

and note that the real solution for y exists if and only if $\xi \geq v(x)$ and that the trajectories are symmetric about the x-axis. It is seen from (12.8) that the slopes are uniquely determined everywhere in xy-plane except at the critical points where both the acceleration (=f(x)) and the velocity are zero. The trajectories in the phase plane can not intersect any where except at the critical points.

Now, we determined the form of trajectories for varies forms of the function v(x).

Case-1:The function v(x) has a maximum:

We consider the case when the energy v(x) has a maximum. When the energy level ξ_0 each level curve consists of two branches which intersect the x-axis and are similar in shape two branches in hyperbola, one opening to the right and the other opening to the left. When $\xi > \xi_0$ each level curve consists of two branches, but in this case they do not intersect the x-axis. When $\xi = \xi_0$, the level curve consists in four branches that meet at branches passing through the saddle point are called the separatrices. None of the other separatrices passes through S and the separatrices are asymptotes to all other trajectories. The critical point S is unstable because any disturbances and more form S and it tends to infinity.

And the infinite amount of time is required by the particle to pass alone a separatrix point to the point itself and this can be seen as follows

We have from (12.10)

$\dot{x} = \pm\sqrt{2\{\xi_0 - v(x)\}}$……………….(12.10A)

Let u = x$-x_0$ which x_0 is the location of the saddle point. Then the nbd of x_0 we have

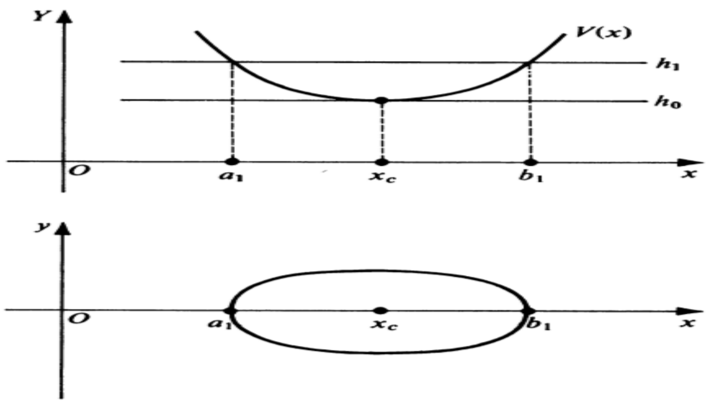
$$\xi_0 - v(x) = \xi_0 - v(x_0)$$

$$\Rightarrow \xi_0 - v(x) = \xi_0 - v(x_0) - uv'(x_0) - \frac{1}{2}u^2 v''(x_0) + O(u^3)$$

$$= -\frac{1}{2}u^2 v''(x_0) + O(u^3) \text{ (Since } v(x_0) = \xi_0 \text{ and } v'(x_0) = 0)$$

Substituting this in (2.10A) and integrating we get the time required to move from $u_1 = x_1 - x_0$ to u = x$-x_0$ as t = $-[-v''(x_0)]^{-\frac{1}{2}}\log\left(\frac{u}{u_1}\right)(x_1 > x_0)$. Since $v(x)$ is maximum at $x_0, so\ u \to 0$ i.e., $x \to x_0\ as\ t \to \infty$.

Case-II: The function $v(x)$ has a minimum:



Suppose v(x) has a minimum at $x = x_0$. When $\xi = \xi_0$, the level curve degeneartes into a single critical point C which is a centre. When $\xi < \xi_0$, there is no real solution, but when $\xi > \xi_0$, each level curve consists of a single closed trajectories surrounding the point C. the critical point C is stable because a small disturbance will a result in a closed trajectories that surrounded C alone which the state of the system remains closed to C, the motion corresponding the closed curves are periodic but need not be harmonic. The period T of the non-linear system is a function of amplitude ξ_0. It can be seen from (12.10A) that

T = $\int_{x_1}^{x_2}[2\{\xi_0 - v(x)\}^{-\frac{1}{2}}]dx$ …………………(12.11)

Near the critical point x_0, we have by putting u = x−x_0

$$\xi_0 - v(x) = \xi_0 - v(u + x_0) = -\frac{1}{2}u^2 v''(x_0) + O(u^3)$$

If the motion is small, then the neglecting the higher order terms in u, the equation of motion $\ddot{x} + f(x) = 0, becomes$

$$\ddot{x} + v'(x) = 0$$

or,$\ddot{u} + \frac{1}{2}v''(x_0).2u\frac{du}{dx} = 0.$

or, $\ddot{u} + v''(x_0).u = 0$

whose solution is

u = $c_1$exp{$\sqrt{-v''(x_0)}$ t} + c_2 exp{−$\sqrt{-v''(x_0)}$ t}

where c_1 and c_2 are constsnts.

Near a centre $v''(x_0) > 0$ and so the centre is oscillatory described by circular functions and so it is stable.

Case-III: The function v(x) has a point of inflection at $x_0 (v'(x) = 0, v''(x_0) = 0)$:

Suppose the maximum and minimum point to form a point of inflection. Each level curve consists of one branch that open to the left. The level curve $\xi = \xi_0$ passes through critical point P which is unstable. The point corresponds to a cusp of the phase plane which can be seen as follows.

At the point P, v(x)=0, $v'(x) = 0, v''(x_0) = 0$.

$\therefore y = 0$, since $\dot{y} = \frac{dy}{dx}\frac{dx}{dt} = \dot{x}\frac{dy}{dx} = y\frac{dy}{dx} = -f(x) = -v'(x) = 0$

$$\therefore \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = -f'(x) = -v''(x) = 0.$$

At x = x_0, y =$\frac{dy}{dx}$ = 0.

Note:

If v(x) or $v'(x)$ is given, we can determine whether the critical point is a saddle point or a centre by examining the second derivative. At the saddle point $v''(x) = f'(x) < 0$ and at a centre $v''(x) = f'(x) < 0$.

For example, consider a equation

$$\ddot{x} + (1 - x)(2 - x) = 0.$$

The critical points are at x = 1 and x = 2.

Now, f(x) = $(1 - x)(2 - x)$

$$\therefore f'(x) = 2x - 3$$

$$f'(1) = -1 < 0 \ and \ f'(2) = 1 > 0$$

Therefore, x = 1 is a saddle point and x = 2 is a centre.

12.3: Parameter Dependent Conservative System:

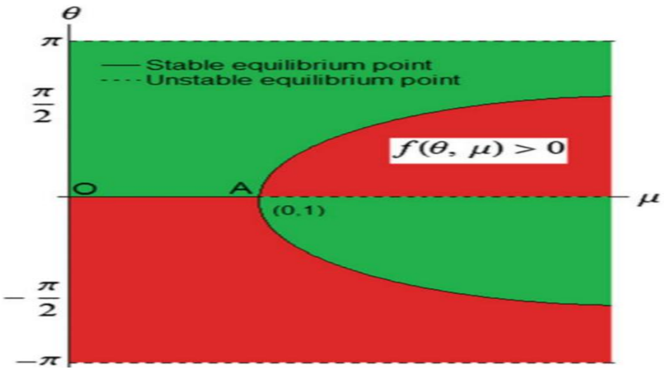
The parameter dependent conservative system is given by the equation

$\ddot{x} + f(x, \lambda) = 0.$ ....(12.12)

where λ is a parameter. The critical point are obtained by the solutions of the equation $f(x, \lambda) = 0$ and so there location depends on the parameter λ. If the potential energy of the system is $V(x, \lambda)$ then

$f(x, \lambda) = \frac{\partial V(x,\lambda)}{\partial x}$ for each λ. ....(12.13)

The critical points corresponds stationary values of the potential energy corresponds to a stable critical point and the other stationary values (maximum and the point of inflation) to be unstable. Infect V is maximum at x $= x_1$if $\frac{\partial V}{\partial x}$ changes from negative to positive on passing through x_1 i.e., $f(x, \lambda)$ changes sign from $+ve\ \ to - ve$ as x increased through x_1.



For example, the solid line between A and B are unstable then C is also unstable, since f is $+ve$ or both sides of C. The nature of the critical point can easily be need from the figure, when $\lambda = \lambda_0$, as shown the system has three critical points two of which are unstable and one is stable. The points A, B and C are known as bifurcation. Points of λ. As λ varies to such points, the critical point may split in two or more or several critical point may appear or marge into a single one.

12.4 Non-Linear Oscillation in Conservative System:

a) **Motion of a Simple Pendulum:**

The equation of motion of a simple pendulum of length l is given by

$\ddot{\theta} + \omega_0^2 sin\theta = 0.$ ....(12.14)

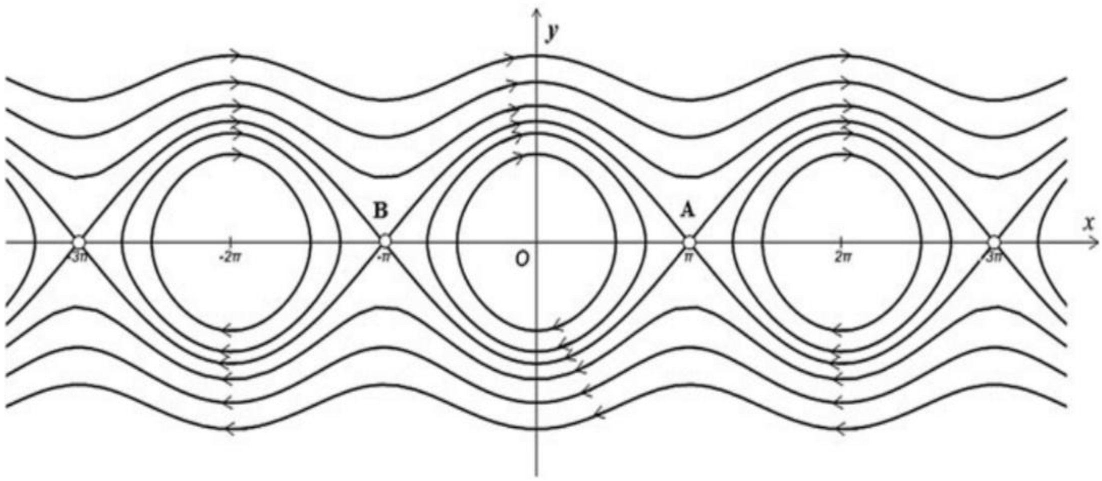
where θ is the angular deviation from two vertical $\omega_0^2 = g/l$. A first integral is $\dot{\theta}^2 = 2[\xi - v(\theta)]$ where $v(\theta) = \omega_0^2 cos\theta$ and ξ, the energy level depends on the initial conditions, we take

$2\,\xi = \dot{\theta}_0^2 - 2\omega_0^2 cos\theta_0$..................(12.15)

So that

$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\omega_0^2\ (cos\theta - cos\theta_0).$ ....(12.16)

Since v(θ) has a minimum $-\omega_0^2$ at $\theta = 0$ and at even multiplies of π, the level curves $\xi = -\omega_0^2$ consists of an infinite number of discrete centre located alone the $\theta - axis$. The centres correspond to stable equilibrium position. Moreover v(θ) has a maximum ω_0^2 at odd multiple of π.



The level curves $\xi = \omega_0^2$ consists of two separatrices shown in figure 2.5 that meet at an infinite number of saddle points located alone the $\theta - axis$ at odd multiples of π. The saddle points correspond to an unstable equilibrium position(invented pendulum). It follows from (12.15) that the equation describing the separatrices is given by

$$\dot{\theta}^2 = 4\omega_0^2 \cos^2\frac{\theta}{2} \text{ (putting } \xi = \omega_0^2)$$

i.e., $\dot{\theta} = \pm 2\omega_0 \cos\frac{\theta}{2}$

When $-\omega_0^2 < \xi < \omega_0^2$, the level curve consists of a infinite number of closed trajectories each of which surrounds one of the centres, they correspond periodic motion of equilibrium position of pendulum. When $\xi > \omega_0^2$ a level curve consists of two way of trajectories outside the separatrices which corresponds to rotating or spinning motion of the pendulum.

From (12.16), we get

$$t = \pm \int_{\theta_0}^{\theta} \left[\dot{\theta}_0^2 + 2\omega_0^2 (cos\theta - cos\theta_0)\right]^{-1/2} d\,\theta$$

For convince, we suppose the motion to be started in the vertical direction ($\theta_0 = 0$) with angular velocity $\dot{\theta}_0$, then we can write

$$t = \pm \frac{1}{|\dot{\theta}_0|} \int_0^{\theta} \frac{d\theta}{\left(1 - k^2 \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}} \quad , \quad where\ k = \frac{2\omega_0}{|\dot{\theta}_0|}$$

The character of the motion varies according to the values k.

If k<1 i.e., $|\dot{\theta}_0| > 2\omega_0$, the integral is always real and the value of θ increases in definitely. In this case $\xi > \omega_0^2$ according to (12.15) and the motion is unbounded and the pendulum undergoes spinning rather than oscillation. The separatrices (12.5) are between trajectories representing the motion of $\theta - axis$.

If k = 1 i.e., $|\dot{\theta}_0| = 2\omega_0$, the integrand is real and approaches to infinite as θ approaches to π.Thus the motion carries the pendulum from straight down to straight up. However θ approaches to π asymptotically as t approaches to ∞. In this case $\xi = \omega_0^2$ according to (12.15) trajectories representing the motion are of the separatrices.

If k>1 i.e., $|\dot{\theta}_0| < 2\omega_0$, the integrand is real only if

$$|\theta| < 2\sin^{-1}\frac{|\dot{\theta}_0|}{2\omega_0} = |\theta_m|(say) \quad \text{..........(12.18)}$$

Thus the pendulum oscillate between $\pm\theta_m$. in this case $-\omega_0^2 < \xi < \omega_0^2$ and the closed trajectories represent this motion. Value k = 1 is called bifurcation value because if separate values of k for which the trajectories vary qualitatively (from open to closed).

In this case the oscillatory motion the integral (12.17) from zero to θ_m with $+ve$ sign, gives 1/4 th period.

Thus the period is

$$T = \frac{1}{|\dot{\theta}_0|}\int_0^{\theta_m}\frac{d\theta}{\left(1-k^2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}, \; k > 1 \quad \text{.............................(12.19)}$$

Let, $k\sin\frac{\theta}{2}\sin\varphi$. Then $\varphi = \frac{\pi}{2}$ when $\theta = \theta_m$ and $k\cos\frac{\theta}{2}.\frac{1}{2}d\theta = \cos\varphi d\varphi$

Then $d\theta = \frac{2\cos\varphi d\varphi}{k\cos\frac{\theta}{2}} = \frac{2k\cos\varphi d\varphi}{(1-k^2\sin^2\varphi)^{\frac{1}{2}}}$ where $k = \frac{|\dot{\theta}_0|}{2\omega_0} = \sin\frac{\theta_m}{2}$

Hence

$$T = \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{(1-k^2\sin^2\varphi)^{\frac{1}{2}}} \quad \text{.......................................(12.20)}$$

This expression for the period is in terms of elliptic function of the first kind (complete normal elliptic integral of first kind).

b) **Motion of Rotating Pendulum:**

Consider a pendulum of mass m and length l constrained to oscillate in a plane rotating with angular velocity Ω about the vertical line. The moment of centrifugal force acting on the pendulum.

$= m\,\Omega^2 a\sin\theta \,.\, a\cos\theta$

$$= m\Omega^2 a^2 \sin\theta\cos\theta$$

and that the gravity mgasin θ

The differential equation of motion is

$$ma^2\ddot{\theta} = m\Omega^2 a^2 \sin\theta \cos\theta - \text{mga}\sin\theta$$

i.e., $\ddot{\theta} = \Omega^2(\cos\theta - \lambda)\sin\theta$ ……………………………………(12.21)

mk^2 is the moment of inertia of mass about the centre.

$\lambda = \frac{g}{a\Omega^2}$, $where \frac{g}{a\Omega^2}$ is a parameter and θ be the angular deviation of the pendulum.

Thus the conservative system is

$\ddot{\theta} + f(\theta,\lambda) = 0$ where $f(\theta,\lambda) = \Omega^2(\lambda - \cos\theta)\sin\theta$

The equivalent system is

$\dot{\theta} = \omega, \omega = \Omega^2(\lambda - \cos\theta)\sin\theta$ ……………………………..(12.22)

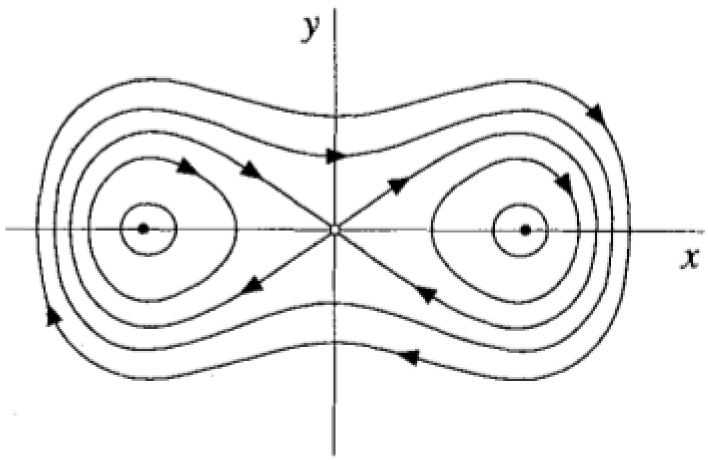
The differential equation of integral curves

$\frac{d\omega}{d\theta} = \frac{\Omega^2(\lambda - \cos\,\theta)\sin\,\theta}{\omega}$ ……………………………(12.23)

The critical points are

$$\theta = 0, \pm\pi, \cos^{-1}\lambda.$$

The corresponding θ,λ diagram is shown in figure-2.7 with remains in which f(θ,λ)>0 shown in shading. The stable and unstable critical points of the diagram are shown by the closed dots and open dots respectively. The former corresponds to the critical point of type centre and later to these the saddle point.



The energy integral in this case is given by

$\omega^2 = \Omega^2[\sin^2\theta + 2\lambda(\cos\theta + 1)]$ ……………………………..(12.24)

In this equation the energy constant is $\xi(=\Omega^2\lambda)$ has been determined by the condition by separatrix passes through the saddle point $\theta = \pm\pi, \dot{\theta} = 0$. As there also exists a second separatrix corresponding to $\theta = 0, \dot{\theta} = 0$ for which the energy constant is $\xi = -\Omega^2\lambda$. We have the relation

$$\omega^2 = \Omega^2[\sin^2\theta + 2\lambda(\cos\theta - 1)] \qquad \text{………………………..(12.25)}$$

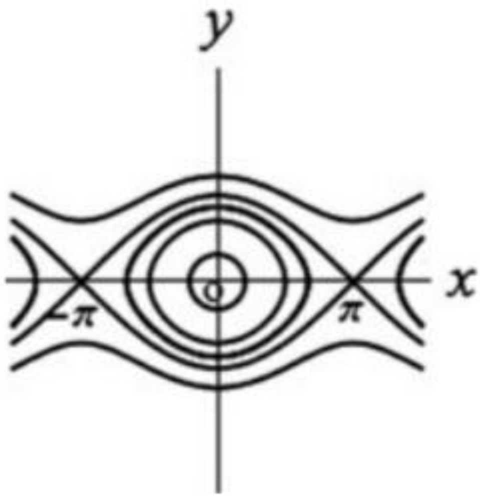


Fig shows in the phase diagram. The differential equation (12.21) with the separatrices the points A(say) to B(say) corresponding to (12.24) and (12.25) respectively.

It is to be noted that the centre at origin for Ω=0 becomes a saddle point for Ω≠ 0 in which case there appear two centresc_1 and c_2 symmetrically placed about the origin. The periodic motion about the centres(within the integral separatrix B are symmetrical). When the energy constant ξ reaches the value corresponding to the separetrix B, the motion changes is character and takes place around two centres c_1 and c_2 and the saddle point S about the origin being still inside the internal separatrix A. in this case, the motion is still oscillation with velocity decreasing in the nbd of $\theta = 0$. If the energy constant is further increased and the separetrix A is crossed.

If $\lambda \to 0$ i.e., Ω→ ∞, the two separatrix A and B approaches each other and the centres c_1 and c_2 approach to the point $\theta = \pm\pi$ respectively.

If $\lambda < 1$, the phase diagram changes again there approach to a centre C, but the intermediate structure of trajectories disappear. If $\lambda = 1$ is the critical or bifurcation value of the parameter.

12.5 Hamiltonian Systems in the Plane:

A system of differential equations on $R^2$is said to be Hamiltonian with one degree of freedom if it can be expressed in the form

$\frac{dx}{dt} = \frac{dH}{dy}, \frac{dy}{dt} = -\frac{dH}{dx}$ ……………………(12.26)

where H(x, y) is a twice differentiable function. The system is said to be conservative and there is no dissipation. The Hamiltonian is defined by

H(x, y) = K (x, y) + V (x, y)

where K is the kinetic energy and V is the potential energy.

Theorem 12.1: Conversation of Energy:

The total energy H(x, y) is first integral and a constant of the motion.

Proof: The total derivative alone a trajectories is given by

$$\frac{dH}{dt} = \frac{dH}{dx}\frac{dx}{dt} + \frac{dH}{dy}\frac{dy}{dt}$$

$= \frac{dH}{dx}\frac{dH}{dy} - \frac{dH}{dy}\frac{dH}{dx} = 0$ (by (12.26))

Thus H(x, y) is a constant alone the solution curves of (12.26) and the trajectories on the Contour‘s are defined by H(x, y)=c, where c is constant.

Definition: A critical point of the system $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in R^2$ at which the Jacobian matrix has non-zero eigen values is called the non-degenerate critical point, otherwise it is called degenerate critical point.

Theorem-12.2:

Any non-degenerate critical point of an analytic Hamiltonian system is either a saddle point or a centre.

Proof:

Assume that the critical point is at the origin. The Jacobian matrix is given by

$$J_0 = \begin{pmatrix} \frac{\partial^2 H(0,0)}{\partial x \partial y} & \frac{\partial^2 H(0,0)}{\partial y^2} \\ -\frac{\partial^2 H(0,0)}{\partial x^2} & -\frac{\partial^2 H(0,0)}{\partial x \partial y} \end{pmatrix}$$

Now, $\det J_0 = \frac{\partial^2 H(0,0)}{\partial x^2}\frac{\partial^2 H(0,0)}{\partial y^2} - \left(\frac{\partial^2 H(0,0)}{\partial x \partial y}\right)^2$

The origin is the saddle point if $\det J_0 < 0$ and the centre or focus if $\det J_0 > 0$.

Note that the critical points of the system (2.26) corresponds to a stationary points of the surface z =H(x, y). If the origin is the focus then the origin is not a strict local maximum or minimum of the Hamiltonian function. Suppose that the origin is a stable focus then $H(x_0, y_0) = \lim_{t\to\infty} H(x(t, x_0, y_0), y(t, x_0, y_0)) = H(0,0)$ for all $(x_0, y_0) \in N_\epsilon(0,0)$ where $N_\epsilon(0,0)$ is a small deleted nbd of the origin. However H(x, y)$>H(0,0)$ at the local minimum and H(x, y)$<H(0,0)$ at the local maximum. A similar point can be applied when the origin is an unstable focus.

Therefore, the non-degenerate critical point is either a saddle point or a centre.

Example 12.2: Find the Hamiltonian for each of the following systems and sketch, the phase path

a) $\dot{x} = y, \dot{y} = x + x^2$
b) $\dot{x} = y + x^2 - y^2, \dot{y} = -x - 2xy$

Solution:

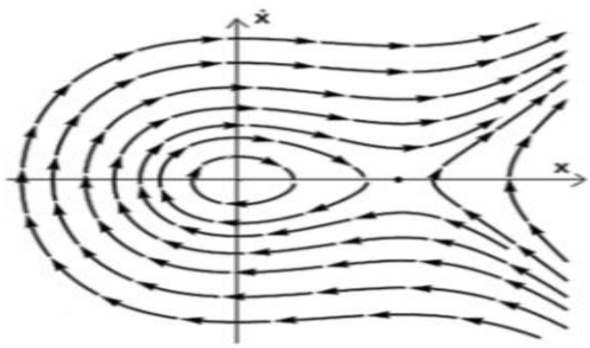
a) Phase path

$J = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

Therefore, (0, 0) is saddle point.

and $J = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 > 0$

Therefore, $(-1, 0)$ is centre.



b) We have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-x - 2xy}{y + x^2 - y^2}$$

or,$ydy + xdx - y^2dy + 2xydx + x^2dy = 0$

Integrating we have,

H(x, y)$= \frac{1}{2}x^2 + \frac{1}{2}y^2 + x^2y - \frac{1}{3}y^3 =$ Constant.

Critical points are at (0, 0), $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ which are all non-degenerate. The critical point at (0, 0) is centre and those are other points are saddle points.

Exercises:

Exercise1: The equation $\ddot{x} + kx + \gamma x^3 = 0, k > 0$ describes the motion of a hard spring if $\gamma > 0$ a soft spring if $\gamma < 0$.

Hint:

Case-1:

If $\gamma > 0$,

The critical point is (0, 0) which is minimum and it is a centre.

Case-2:

If $\gamma < 0$,

The critical point are at (0, 0) and $(\pm\sqrt{-\frac{k}{\gamma}}, 0)$, (0, 0) is minimum and it is a centre. $(\pm\sqrt{-\frac{k}{\gamma}}, 0)$ is maximum and it is saddle point.

Exercise2: Find the equation of the path $\ddot{x} - x + 2x^3 = 0$ and sketch of the path in the phase-plane. Locate the critical points and determine the nature of each.

Hint:

The critical points

(0, 0) is maximum and it is saddle point and $(\pm\frac{1}{2}, 0)$ is minimum and it is a centre.

Exercise3: For each of the following systems, sketch the solution trajectories in the phase plane and the indicate on the sketch there critical points and their types as well as separatrices:

i) $\ddot{u} + u - 2u^3 = 0$
ii) $\ddot{u} - u + u^3 = 0$
iii) $\ddot{u} + u + u^3 = 0$
iv) $\ddot{u} - u - u^3 = 0$

v) $\ddot{u} + u^3 = 0$

vi) $\ddot{u} + u - \frac{\lambda}{a-u} = 0$

Hints:

(i) and (iii) → One centre and two saddle points.

(ii) and (iv) → Two centre and one saddle points.

(v) → One centre.

(vi) → $if\ \lambda < 0$, then it has two centre.

$if\ \lambda = 0$, then it has one centre.

if $0 < \lambda < \frac{a^2}{4}$, then it has one saddle point and centre.

if $\lambda > \frac{a^2}{4}$, then it represents no critical point.

UNIT-13

Limit Cycles: Poincaré-Bendixon theorem (statement only). Criterion for the existence of limit cycle for Liénard's equation.

13.1 Introduction:

We have already encountered autonomous systems having closed paths. For example, the system has a center at (0,0) and in the neighborhood of this center there is an infinite family of closed paths resembling ellipses (see Figure 13.21). In this example the closed paths about (0,0) form a continuous family in the sense that arbitrarily near to any one of the closed paths of this family there is always another closed path of the family. Now we shall consider systems having closed paths which are isolated in the sense that there are no other closed paths of the system arbitrarily near to a given closed path of the system. What is the significance of a closed path? Given an autonomous system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{13.1}$$

one is often most interested in determining the existence of periodic solutions of this system. It is easy to see that periodic solutions and closed paths of are very closely related. For, in the first place, if $x = f_1(t), y = g_1(t)$, where f_1 and g_1 are not both constant functions, is a periodic solution of (13.4), then the path which this solution defines is a closed path. On the other hand, let C be a closed path defined by a solution $x = f(t), y = g(t)$, and suppose $f(t_0) = x_0, g(t_0) = y_0$. Since C is closed, there exists a value $t_1 = t_0 + T$, where $T > 0$, such that $f(t_1) = x_0, g(t_1) = y_0$. Now the pair $x = f(t+T), y = g(t+T)$ is also a solution of (13.4). At $t = t_0$, this latter solution also assumes the values $x = x_0, y = y_0$. the two solutions $x = f(t), y = g(t)$ and $x = f(t+T), y = g(t+T)$ are identical for all t. In other words, $f(t+T) = f(t), g(t+T) = g(t)$ for all t, and so the solution $x = f(t), y = g(t)$ defining the closed path C is a periodic solution. Thus, the search for periodic solutions falls back on the search for closed paths.Now suppose the system has a closed path C. Further, suppose it possesses a nonclosed path C_1 defined by a solution $x = f(t), y = g(t)$ and having the following property: As a point R traces out C_1 according to the equations $x =$

$f(t)$, $y = g(t)$, the path C_1 spirals and the distance between R and the nearest point on the closed path C approaches zero either as $t \to +\infty$ or as $t \to -\infty$. In other words, the nonclosed path C_1 spirals closer and closer around the closed path C either from the C_1 approaches C from the outside).
In such a case we call the closed path C a limit cycle, according to the following definition:

13.2 (Definition):

Poincare showed that the differential equation of the form

$\dot{x} = x(x,y), \dot{y} = y(x,y)$ ……………………..(13.2)

Admits occasionally special solutions represented by the closed curves in the phase-plane which called limit cycles. A limit cycle is called closed trajectory (hence the trajectory of periodic solution) such that no trajectory sufficiently near to it is closed. In other words a limit cycle is an isolated closed trajectory. Every trajectory beginning sufficiently near a limit cycle approach it for $t \to \infty$ or for $t \to -\infty$ i.e., it either winds itself upon the line cycle or unwinds form it. In all nearby trajectories approach a limit cycle C as $t \to \infty$, we say that C is stable (figure 13.1a), if they approach C as $t \to -\infty$, we say that that C is unstable(figure 13.1b). It is trajectories on one side of C approach it while those on the other side depart from it, we say that C is semi-stable(figure 13.1c).

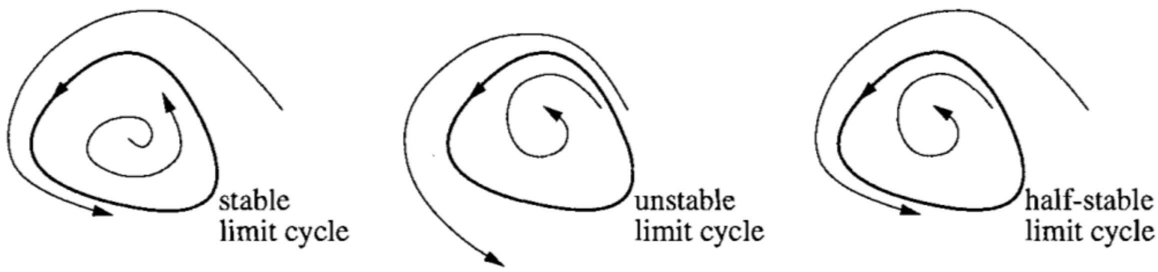


Figure 13.1a Figure 13.1b Figure 13.1c

Stable limit cycle unstable limit cycle semi-stable limit cycle

$(t \to \infty)$ $(t \to -\infty)$ $(\to \infty$ and $t \to -\infty)$

13.2 Example of Limit Cycles:

a) Consider the system

$$\dot{x} = y + \frac{x}{\sqrt{x^2+y^2}}\{1-(x^2+y^2)\}$$

$$\dot{y} = -x + \frac{y}{\sqrt{x^2+y^2}}\{1-(x^2+y^2)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13.3)$$

It polar co-ordinates it becomes,

$\dot{x} = y + \frac{x}{r}(1 - r^2)$, $\dot{y} = -x + \frac{y}{r}(1 - r^2)$ (Since x = rcosθ, y = rsinθ)

Noting that, $x\dot{x} + y\dot{y} = \frac{1}{2}\frac{d}{dt}(r^2)$ and $\dot{x}y - \dot{y}x = -r^2\dot{\theta}$ $(\dot{x} = \dot{x}\cos\theta - rsin\dot{\theta}$ and $\dot{y} = \dot{r}\sin\theta + rcos\,\theta.\dot{\theta})$

We get,

$$\frac{1}{2}\frac{d}{dt}(r^2) = x\{y + \frac{x}{r}(1 - r^2) + y\{-x + \frac{y}{r}(1 - r^2)\}$$

or, $r\dot{r}$ = r$(1 - r^2)$

and$-r^2\dot{\theta} = y\{y + \frac{x}{r}(1 - r^2) + x\left\{-x + \frac{y}{r}(1 - r^2)\right\} = (x^2 + y^2) = r^2$.

i.e.,$\dot{\theta} = -1 = Constant.$

Since $\dot{\theta} = Constant$, so the radius vector with constant angular velocity. The equation are $\dot{r}$ = $(1 - r^2)$on integration leads to

$r = \frac{Ae^{2t}-1}{Ae^{2t}+1}$

Initially, t = 0 then $r = r_0 = \frac{A-1}{A+1}$ where $r_0(\neq 1)$ being the initial value of r.

Now $r \to 1$ as $t \to \infty$ and the limit cycle in the case is a circle of radius unity. If (r_0>1) the spiral winds itself onto the circle. If r = 1 from the inside. The limit cycle in this case is stable.

b) Consider the differential equation

$$\dot{x} = -y + x(x^2 + y^2 - 1)$$

$$\dot{y} = x + y(x^2 + y^2 - 1) \qquad \text{..........................(13.3)}$$

The polar equations those equation given by

$\dot{r}$ = $r(1 - r^2), \dot{\theta} = 1$.

The equation $\dot{r}$ = $r(1 - r^2)$ tends to the solution

$$r^2 = \frac{1}{1 - Ae^{2t}}$$

When $t \to \infty$ then r = 0 i.e., the path does not exists.

Thus $r \to 1$ as $t \to -\infty$. Hence the limit cycle r = 1 is unstable.

(c) The differential equations

$$\dot{x} = y + x\sqrt{x^2 + y^2}(x^2 + y^2 - 1)^2$$

$$\dot{y} = -x + y\sqrt{x^2+y^2}(x^2+y^2-1)^2$$

Gives an example of semi-stable limit circle. In the polar co-ordinates the system reduces to

$\dot{r} = r^2(1-r^2)^2, \dot{\theta} = -1.$

and the equation of first two differential equation leads to the required limit cycle.

Note: The phase portrait for the equation tells the story. The equilibrium point r=0 is a source whereas r=1 is a node because $\dot{r}$>0 for 0<r<1 and r>1 as well. The graphical interpretation of this fact is that the unit circle described by r=1 is a semi-stable limit cycle. Trajectories approach the unit circle from inside it whereas trajectories that start outside escape the unit circle.

13.3Negative Criterion of Bendixon:

Bendixon establish a theorem for the non-existence of the limit cycles and this theorem is known as negative criterion of Bendixon and given a sufficient condition.

Theorem-13.1: Given a system of differential equations $\dot{x} = X(x,y), \dot{y} = Y(x,y)$; the negative condition of Bendixon states that if the expansion $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$ does not change its sign(or vanish identically) within a region D of the phase plane R^2, no closed trajectory can exists in D (where D is simple connected domain).

Proof:

By Green's theorem we have,

$$\iint_D \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) dxdy = \oint_C (Xdy - Ydx)$$

In the contour C even which the integration is performed in a closed trajectory of the equations, the line integral

$$\oint_C (Xdy - Ydx)$$

$$= \oint_C (X\dot{y} - Y\dot{x})\, dt$$

$$= \oint_C (\dot{x}\dot{y} - \dot{y}\dot{x})\, dt = 0.$$

This contradicts the hypothesis according to which the double integral can not vanish which implies if $(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y})$ does not vanish or change it sign, then no closed path exists in D.

13.4: Poincare-Bendixon Theorem (P-B Theorem):

Statement: Let D be the bounded region of phase-plane together with its boundary and assume that D does not contain critical point of the system $\dot{x} = X(x, y), \dot{y} = Y(x, y)$. If C = [x(t), y(t)] is the path of the given system that lies in D for all $t \geq t_0$, then C is either a closed path or spirals towards a closed path as $t \to \infty$. Thus in system has a closed path in D.

Example-1:

a) Show that the following non-linear autonomous system $\dot{x} = 4x + 4y - x(x^2 + y^2), \dot{y} = 4x + 4y - y(x^2 + y^2)$ has a periodic solution.
b) Show that the equation $\ddot{x} + f(x)\dot{x} + g(x)x = 0$ can have no periodic solution whose path lies in a region where f is of one sign(Applied negative criteria of Bendixon).

13.5Lienard'sEquation: The equation of the form

$\ddot{x} + f(x)\dot{x} + g(x)x = 0$ …………………………(13.5)

is generally known as Lienard's equation. It is a suppose that f(x) is a positive when $|x|$ is a large and negative and $|x|$ is small, and g(x) is such that in the absence of the damping term f(x)$\dot{x}$ we exposed periodic solutions for small x.

Let us put $\dot{x} = y - F(x), \dot{y} = -g(x)$ ………………………..(13.6)

where F(x)= $\int_0^x f(u)du$

Statement of Lienard's Theorem:

The equation $\ddot{x} + f(x)\dot{x} + g(x)x = 0$ has a unique periodic solution if f(x) and g(x) are continuous and

(i) F(x) is an odd function.
(ii) F(x) = 0 only at x = 0 and x = $\pm a$ for some a > 0.
(iii) F(x)→ ∞ as x→ ∞ monotonically for x > a.
(iv) g(x) is an odd function.

Note-1: The unique periodic solution corresponds to a unique closed path surrounding to origin in the phase-plane and this path is approached spirally (by every other path) as $t \to \infty$.

Note-2: The general pattern of the path can be obtained from the following consideration.

(a) If [x(t), y(t)] is a solution, so is [−x(t), −y(t)] (since F(x) and g(x) are odd). Thus the phase diagram is symmetrical about the origin.
(b) The slope of the path is given by
$\frac{dy}{dx} = \frac{-g(x)}{y-F(x)}$ ……………………………….(13.7)
so the path are horizontal only on x = 0(by(iv)) and vertical only on the curve y = F(x). About y = F(x), $\dot{x} > 0$ and below $\dot{x} < 0$.

(c) $y > 0 \; if \; x > 0 \; and \; y > 0 \; if \; x < 0\big(by\ (iv)\big)$.

Lienard's Criterion for the Existence of Limit Cycle:

Let $v(x,y) = \int_0^x f(u)du + \frac{1}{2}y^2$ ………………………….(13.8)

where $\frac{1}{2}y^2$ may be regarded as K.E and G(x) = $\int_0^x f(u)du$ as the potential energy so that $v(x,y)$ is the total energy stored in the oscillation. Alone the element of the path we have,

$$dv = gdx + ydy = g(x)\frac{dx}{dy}dy + ydy = g(x)\frac{y-F(x)}{-g(x)}dy + ydx = F(x)dy$$

The energy exchange of the system is $\int F(x)dy$ and if the system is in a stationary state of the oscillation alone the closed path C then we have,

$\oint F(x)dy = 0$………………………………(13.9)

This linear integral is to be taken alone a trajectory. Equation (13.9) is Lienard's Criteria for the existence of the limit cycle for Lienard's equations (13.5).

13.6 Lienard's Method of Constructing Integral Curves.

Consider the equation

$\ddot{x} + f(\dot{x}) + \omega^2 x = 0$ ………………..(13.10)

Let $\tau = \omega t$ and then the equation (13.10) transformed into

$x'' + \varphi(x') + \omega^2 x = 0$ ……………………….(13.11)

where prime indicates derivative w.r.to τ and $\varphi(x') = \frac{1}{\omega} f(\omega x^2)$.

Putting $x' = y, y' = -\varphi(y) - x$

We have the following differential equations for the trajectories

$\frac{dy}{dx} = -\frac{\varphi(y)+x}{y}$ ………………….(13.12)

To draw the trajectories, we first plot the curve $x = -\varphi(y)$ on the phase plane.

To initiate the trajectory passing through point A, we draw the line parallel to the x-axis intersecting the curve $x = -\varphi(y)$ at the point B. Construct the line BC parallel to the y-axis, intersecting the x-axis at the point C. Then the line CA is perpendicular to the direction field at A, because the slope of CA is

$\frac{BC}{AB} = \frac{y}{x+\varphi(y)}$. We draw a line from A perpendicular to AC and approximate the integral by the short line segment AA, alone the direction field. Then starting with A_1, we repeat the process.

13.7 Asymptotic Cases of Lienard's Equation:

We consider Lienard's equation with parameter μ in the form

$\ddot{x} + \mu f(x)\dot{x} + x = 0$ ……………………………(13.13)

and we pass to the asymptotic case $\mu \to 0$ *and* $\mu \to \infty$. Let us put $\dot{x} = y - \mu F(x)$ *and* $\dot{y} = -x$ where F(x) = $\int_0^x f(u)du$. Then we have

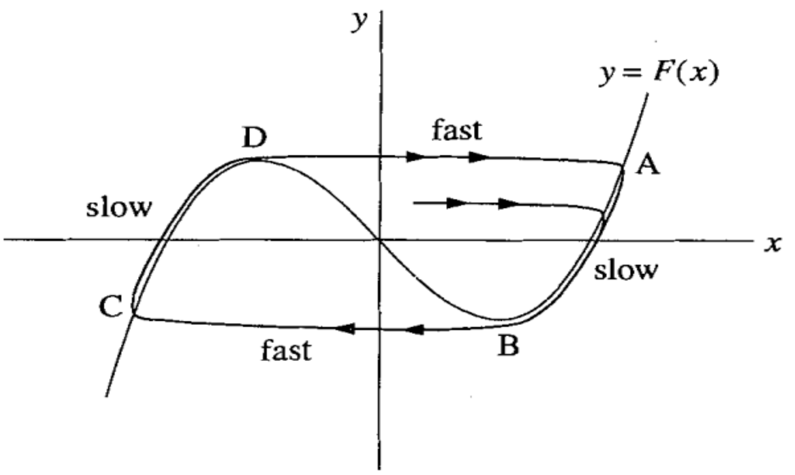
$\frac{dy}{dx} + \frac{x}{y-\mu F(x)} = 0$ ……………………….(13.14)

Let y = μz, then the equation (13.14) is transformed into

$\frac{dz}{dx} = -\frac{x}{\mu^2[z-F(x)]}$ …………………………(13.15)

If $\mu > 1$, for some x, the integral curves are smaller slopes than previously. If $\mu \to \infty$ then $\frac{dz}{dx} \to 0$ thus for increasing μ the integral curves exhibit flat portions parallel to the x-axis.

In the asymptotic case $\mu \to \infty$ then equation (13.15) reduces to $[z - F(x)]$dz= 0. This suggest that the integral curve consists of two branches : on one of them there exists the relation. z = F(x) and on the other dz = 0 i.e., this branch the straight line parallel to the x-axis.



In other to investigate the velocity of the representative point R(x(t), y(t)), we have

$\dot{x} = \mu[z - F(x)]$ and $\dot{z} = -\frac{x}{\mu}$ ……………(13.16)

In R follows the branch z = F(x), in the asymptotic case when μ is large, the velocity $\dot{x}$ is finite. For the second branch z $\neq F(x)$, $\dot{x}$ is large. Thus the horizontal branches (z = constant) are traversed with a very high velocity, where the characteristic F(x) is traversed with a finite velocity. This gives rise to a situation shown in figure 3.3 corresponding to f(x) $=x^2 - 1$ i.e., F(x)= $\frac{1}{3}x^3 - x$, as in the case of Van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$.

The point R follows F(x) upto the point where the second branch z = constant. On this branch BC, the point acquired a very high velocity and practically in no time. At C begins again the first branch traversed with finite velocity upto the point D where another jump DA begins thus closing, the curve BCDAB consisting of two distinct branches.

The second asymptotic case i.e., when μ is small, is less integrating. Here Lienard's equation can be written as

$$xdx + ydy - \mu F(x)dy = 0.$$

which reduces to $xdx + ydy = 0$ when $\mu \to 0$ which gives a $x^2 + y^2 = constant$ family of concentric circle with centre at origin.

Definition 1:

Let C be a path of the system (13.1) and let x = f(t), y = g (t) be a solution of (13.1) defining C. Then we shall call set of all points of C for t$\geq t_0$, where t_0 is some value of t, a half-path of (13.1). In otherwords, by a half-path of (13.1) we mean the set of all points with co-ordinates [f(t), g(t)] for $t_0 \leq t < +\infty$. We denote a half-path of (13.1) by C^+.

DEFINITION 2:

Let C^+be a half-path of (13.4) defined by $x = f(t), y = g(t)$ for $t \geq t_0$. Let (x_1, y_1) be a point in the xy plane. If there exists a sequence of real numbers $\{t_n\}, n = 1,2, \ldots$, such that $t_n \to +\infty$ and $[f(t_e)g(t_n)] \to (x_1, y_1)$ as $n \to +\infty$, then we call $(x_1, y_1)a$ limit point of C^+. The set of all limit points of a half-path C^+will be called the limit set of $C^+$and will be denoted by $L(C^*)$.

Example:

The paths of the system (13.3) are given by Equations . Letting $c = 1$ we obtain the path C defined by

$$x = \frac{\cos t}{\sqrt{1 + e^{-2t}}},$$

$$y = -\frac{\sin t}{\sqrt{1 + e^{-2t}}},$$

The set of all points of C for $t \geq 0$ is a half-path C^+. That is, $C^+$is the set of all points with coordinates

$$\left[\frac{\cos t}{\sqrt{1+e^{-2t}}}, -\frac{\sin t}{\sqrt{1+e^{-2t}}}\right], \quad 0 \leq t < +\infty.$$

Consider the sequence $0, 2\pi, 4\pi, \ldots, 2n\pi, \ldots$, tending to $+\infty$ as $n \to +\infty$. The corresponding sequence of points on $C^+$is

$$\left[\frac{\cos 2n\pi}{\sqrt{1+e^{-4n\pi}}}, -\frac{\sin 2n\pi}{\sqrt{1+e^{-4n\pi}}}\right], \quad (n = 0,1,2,\ldots),$$

and this sequence approaches the point (1,0) as $n \to +\infty$.
Thus (1,0) is a limit point of the half-path C^*.
The set of all limit points of $C^+$is the set of points such that $x^2 + y^2 = 1$. In other words, the circle $x^2 + y^2 = 1$ is the limit set of C^+.
We are now in a position to state the Poincare-Bendixson theorem.

THEOREM 13.1 Poincare-Bendixson Theorem; "Strong"' Form:

Hypothesis

1 Consider the autonomous system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{13.1}$$

where P and Q have continuous first partial derivatives in a domain D of the xy plane. Let D_1 be a bounded subdomain of D, and let R denote D_1 phus its boundary.

2 Let C^+defined by $x = f(t), y = g(t), t \geq t_0$, be a half-path of $(I3,1)$ contained entirely in R. Suppose the limit set $L(C^+)$of C^+contains no critical points of (13.1). Conclusion. Either (l) the half-path $C^+$is itself a closed path [in this case $C^+$and $L(C^+)$are identical], ar (2)$L(C^+)$is a closed path which C^+approaches spirally from either the inside or the outside [in this case $L(\mathrm{C}^+)$is a limit cycle]. Thus in either case, there exists a closed path of (13.1) in R.

A slightly weaker but somewhat more practical form of this theorem may be seen at once. If the region R of Hypothesis 1 contains no critical points of (13.4), then the limit set $L(C^+)$will contain no critical points of (13.1) and so the second statement of Hypothesis 2 will automatically be satisfied. Thus we may state:

THEOREM 13.1A Poincaré-Bendixson Theorem; "Weak" Form

Hypothesis

1 Exactly as in Theorem 13.I.

2 Suppose R contains no critical points of (13.1).
Conclusion, If R contains a half-path of (13.1), then R also contains a closed path of (13.1).

Let us indicate how this theorem may be applied to determine the existence of a closed path of (13.1). Suppose the continuity requirements concerning the derivatives of $P(x,y)$ and $Q(x,y)$ are satisfied for all (x,y). Further suppose that (13.1) has a critical point at (x_0,y_0) but no other critical points within some circle

$$K=(x-x_0)^2+(y-y_0)^2=r^2$$

about (x_0,y_0) (see Figure 13.31). Then an annular region whose boundary consists of two smaller circles $K_1:(x-x_0)^2+(y-y_0)^2=r_1^2$ and $K_2:(x-x_0)^2+(y-y_0)^2=r_2^2$, where $0<r_1<r_2<r$, about (x_0,y_0) may be taken as a region R containing no critical points of (13.1). If we can then show that a half-path $C^+$of (13.1) (for $t\geq$ some τ_0) is entirely contained in this annular region R, then we can conclude at once that a closed path C_0 of (13.1) is also contained in R.

The difficulty in applying Theorem 13.1A usually comes in being able to show that a half-path $C^+$is entirely contained in R. If one can show that the vector $[P(x,y),Q(x,y)]$ determined by (13.1) points into R at ever y point of the boundary of R, then a path C entering R at $t=t_0$ will remain in R for $t\geq t_0$ and hence provide the necded half-path C^+.

Example:
Consider again the system (13.3) with critical point (0,0). The annular region R bounded by $x^2+y^2=\frac{1}{4}$ and $x^2+y^2=4$ contains no critical points of (13.3). If we can show that R contains a half-path of (13.3), the Poincaré-Bendixson theorem ("weak" form) will apply.
In our previous study of this system we found that

$$\frac{dr}{dt}=r(1-r^2)$$

for $r>0$, where $r=\sqrt{x^2+y^2}$. On the circle $x^2+y^2=\frac{1}{4}$, $dr/dt>0$ and hence $r=\sqrt{x^2+y^2}$ is increasing. Thus the vector $[P(x,y),Q(x,y)]$ points into R at every point of this inner circle. On the circle $x^2+y^2=4, dr/dt<0$ and hence $r=\sqrt{x^2+y^2}$ is decreasing. Thus the vector $[P(x,y),Q(x,y)]$ also points into R at every point of this outer circle. Hence a path C entering R at $t=t_0$ will remain in R for $t\geq t_0$, and this provides us with the needed half-path contained in R.

Thus by the Poincark-Bendixson theorem ("weak" form), we know that R contains a closed path C_0 - We have already seen that the circle $x^2+y^2=1$ is indeed such a closed path of (13.3).

13.6 The Index of a Critical Point:

We again consider the system (13.1), where P and Q have continuous first partial derivatives for all (x, y), and assume that all of the critical points of (13.1) are isolated. Now consider a simple closed curve* C[not necessarily a path of (13.1)] which passes

13.7 The Lienard-Levinson-Smith Theorem and the van der Pol Equation:

Throughout this chapter we have stated a number of important results without proof. We have done this because we feel that every serious student of differential equations should become aware of these results as soon as possible, even though their proofs are definitely beyond our scope and properly belong to a more advanced study of our subject. In keeping with this philosophy, we close this section by stating without proof an important theorem dealing with the existence of periodic solutions for a class of second-order nonlinear equations. We shall then apply this theorem to the famous van der Pol equation already introduced at the beginning of the chapter.

THEOREM 13.2 Lienard-Levinson-Smith:

Hypothesis. Consider the differential equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \qquad (13.17)$$

where f, g, F defined by $F(x) = \int_0^x f(u)du$, and G defined by $G(x) = \int_0^x g(u)du$ are real functions having the following properties:

1 f is even and is continuous for all x.

2 There exists a number $x_0 > 0$ such that $F(x) < 0$ for $0 < x < x_0$ and $F(x) > 0$ and monotonic increasing for $x > x_0$. Further, $F(x) \Rightarrow \infty$ as $x \to \infty$.

3 g is add, has a continuous deribative for all x, and is such that $g(x) > 0$ for $x > 0$.

4 $G(x) \to \infty$ as $x \to \infty$.
Conclusion. Equation (13.17) possesses an essentially unique nontrivial periodic solution.

Remark: By "essentially unique" in the above conclusion we mean that if $x = \phi(t)$ is a nontrivial periodic solution of (13.17), then all other nontrivial periodic solutions of (13.17) are of the form $x = \phi(t - t_1)$, where t_1 is a real number. In other words, the equivalent autonomous system

$$\frac{dx}{dt} = y, \qquad\qquad (13.18)$$
$$\frac{dy}{dt} = -f(x)y - g(x),$$

has a unique closed path in the xy plane.
One of the most important examples of an equation of the form (13.17) which satisfies the hypotheses of Theorem 13.2 is the van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0, \qquad (13.19)$$

where μ is a positive constant. Here $f(x) = \mu(x^2 - 1), g(x) = x$,

$$F(x) = \int_0^x \mu(u^2 - 1)du = \mu\left(\frac{x^3}{3} - x\right).$$

and

$$G(x) = \int_0^x udu = \frac{x^2}{2}.$$

We check that the hypotheses of Theorem 13.2 are indeed satisfied:

1 Since $f(-x) = \mu(x^2 - 1) = f(x)$, the function f is even. Clearly it is continuous for all x.

2 $F(x) = \mu(x^3/3 - x)$ is negative for $0 < x < \sqrt{3}$. For $x > \sqrt{3}, F(x)$ is positive and monotonic increasing (it is, in fact, monotonic increasing for $x > 1$). Clearly $F(x) \to \infty$ as $x \to \infty$.

3 Since $g(-x) = -x = -g(x)$, the function g is odd. Since $g'(x) - 1$, the derivative of g is continuous for all x. Obviously $g(x) > 0$ for $x > 0$.

4 Obviously $G(x) \to \infty$ as $x \to \infty$.
Thus the conclusion of Theorem 13.2 is valid, and we conclude that the van der Pol equation (13.18) has an essentially unique nontrivial periodic solution. In other words, the equivalent autonomous system

$$\frac{dx}{dt} = y, \qquad\qquad (13.20)$$
$$\frac{dy}{dt} = \mu(1 - x^2)y - x,$$

has a unique closed path in the xy plane.

The differential equation of the paths of the system (13.93) is

$$\frac{dy}{dx} = \frac{\mu(1 - x^2)y - x}{y}.$$

Using the method of isoclines one can obtain the paths defined by (13.20) in the xy plane. The results for $\mu = 0.1, \mu = 1$, and $\mu = 10$ are shown.respectively. The limit cycle C in each of these figures is the unique dosed path whose existence we have already ascertained on the basis of Theorem 13.2. For $\mu = 0.1$ we note that this limit cycle is very nearly a circle of radius 2 . For $\mu = 1$, it has lost its circle-like form and appears rather "baggy," while for $\mu = 10$, it is very long and narrow.

Exercises:

(i) Show that the equation
$\ddot{x} + \frac{x^2+|x|-1}{x^2-|x|+1}\dot{x} + x^3 = 0$ has a unique period solution.

(ii) Show that the Van der Pol equation
equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, (\mu > 0)$ has a unique periodic solution(apply Lienard's theorem).

Hints:x> 0, F(x) = $\int_0^x f(u)du = x + \log(x^2 - x + 1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} - \frac{\pi}{3\sqrt{3}}$

and if **,** x < 0, F(x) = $x - \log(x^2 + x + 1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \frac{\pi}{3\sqrt{3}}$.

UNIT-14

Stability: Definition in Liapunov sense. Routh-Hurwitz criterion for nonlinear systems.

14.1 Introduction:

The equation of stability is concerned with what happens a system is disturbed near an equilibrium condition in general terms near an unstable equilibrium condition leads to a larger and larger departure from this condition. Near a stable equilibrium condition, the opposite is the case and the equilibrium condition may be either stationary or oscillatory. When it is stationary, the variables of the system remain constant and when it is oscillatory, the variable undergo continuous periodic change. The stability of a linear system is well defined but since new type of phenomenon arises in a non-linear system, it is not possible to use the single definition of stability which is meaningful on all cases. For this reason, stability is defined in a number of ways.

14.2 Stability of Equilibrium Solutions (Liapunor Stability):

Consider the regular system

$$\dot{\vec{x}}= \vec{f}(\vec{x},t), \vec{x} \in D \subset R^n, t \in R \text{ ....(14.1)}$$

with $\vec{f}(\vec{x},t)$ continuous in $\vec{x}$ and t and Lipchitz continuous in $\vec{x}$.

Definition-1: Liapunor Stability:

Let $\vec{x}(t)$ be a given regular complex solution of (4.1). Then $\vec{x}(t)$ is said to be Liapunor stability for $t \geq 0$ if for any $\epsilon > 0$, then there exist a $\delta(t,t_0) > 0$ such that

$$\|\vec{x}(t_0) - \vec{x}^*(t_0))\| < \delta \Rightarrow \|\vec{x}(t) - \vec{x}^*(t))\| < \epsilon \; for\; all\; t \geq t_0 \text{............(14.2)}$$

where $\vec{x}(t)$ is any other solution. Otherwise $\vec{x}(t)$ is said to be unstable.

Definition-2: Uniform Stability:

If a solution is stable for $t \geq t_0$ and the δ ofdefinition-1 is dependent of t_0, the solution is said to be uniformly stable on $t \geq t_0$.

It is clear that any stable solution of an autonomous system must be uniformly stable, since the stable is invariant w,r,t time translation.

Definition-3: Asymptotic Stability:

Let $\vec{x}(t)$ be stable (or uniformly stable) solution for $t \geq t_0$. If additionally there exists $\eta(t_0) > 0$ such that

$$\|\vec{x}(t_0) - \vec{x}^*(t_0))\| < \boldsymbol{\eta}$$

$\lim_{t\to\infty} \ \|\vec{x}(t) - \vec{x}^*(t))\| = 0.$ ………..(14.3)

Then the solution is said to be asymptotic stable (or uniformly asymptotically stable).

14.3: Stability of Periodic Solutions:

Definition-4: Liapunor Stability:

Consider the equation$\dot{\vec{x}}= \vec{f}(\vec{x}, t)$ with periodic solution $\vec{\varphi}(t)$. The periodic solution is Liapunor stable if for each t_0 and $\epsilon > 0$ we can find $\delta(\epsilon, t_0)$ such that

$$\|\vec{x}(t_0) - \vec{\varphi}(t_0))\| < \boldsymbol{\delta}$$

$\Rightarrow \|\vec{x}(t) - \vec{\varphi}(t))\| < \epsilon$, for all $t \geq t_0$.

Definition-5:Poincare Stability:

Let C be a closed orbit (a closed path) of $\dot{\vec{x}}= \vec{f}(\vec{x})$. We say that C is periodically or orbitally stable if given any $\epsilon > 0$, we can find $\delta(\epsilon)$ such that if R is a representative point of another trajectory which is within a distance δ of C at a time t_0, then R remains within a distance ϵ of C for all $t \geq t_0$. If no such δ exists, C is said to be periodically or orbitally unstable.

Let C be the orbitally unstable. If, in addition, the distance between R and C tends to zero as $t \to \infty$, it is said to be asymptotically periodically or asymptotically orbitally stable.

14.4: Linear Equations:

There is a large numbers of theorems of linear equations of which we state a summary of some important results.

I. **Equations with Constant Coefficients:**
Consider the equation
$\dot{\vec{x}} = A\vec{x}$ ……………..………….(14.4)

with A, a non-singular constant nxn matrix. The eigen values are solutions of the characteristic equations

$\det(A-\lambda I) = 0.$ ……………………….(14.5)

Suppose that the eigen values λ_k are distinct with corresponding eigen vectors $\overrightarrow{C_k}$ (k = 1, 2, …, n). In this case $\overrightarrow{C_k}e^{\lambda_k t}$(k = 1, 2, …, n) are n-independent solutions (14.4). Suppose now that not an eigen values are distinct, for instance the eigen value $\boldsymbol{\lambda}$ multiplicity m>1. This eigen value $\boldsymbol{\lambda}$ generates m-independent solutions of the form $\overrightarrow{P_0}(t)e^{\lambda t}$, $\overrightarrow{P_1}(t)e^{\lambda t}$, …, $\vec{P}_{m-1}(t)e^{\lambda t}$ where $\vec{P}_k(t)$, (k = 1, 2, …, m-1) of polynomial vectors of degree k.

We compose n-independent solutions $x_1(t), x_2(t), \dots, x_n(t)$ of equations (14.4) to from a matrix $\varphi(t)$ with the solutions as columns

$$\varphi(t) = \left(x_1(t), x_2(t), \dots, x_n(t)\right)^T$$

where $\varphi(t)$ is called a fundamental matrix of equation (14.4). Each solution of equation (4.4) can be written as

$$\vec{x}(t) = \varphi(t)\overrightarrow{C_k}\ .$$

where $\overrightarrow{C_k}$ is constant vector using the initial condition $\vec{x}(t_0) = \vec{x}_0$, we have the required solution

$$\vec{x}(t) = \varphi(t)\varphi^{-1}(t_0)\vec{x}_0 \quad \text{…………………………..(14.6)}$$

We may choose the fundamental matrix $\varphi(t)$ such that $\varphi(t_0) = I$, the nxn identity matrix.

Theorem-14.1:

Consider the equation $\dot{\vec{x}} = A\vec{x}$ with A non-singular constant nxn matrix havingeigen values $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \dots, \boldsymbol{\lambda}_n$.

(a) If Real $\boldsymbol{\lambda}_k<0$, (k = 0, 1, 2,…,n), then for each$\vec{x}(t_0) = \vec{x}_0 \in R^n$ and suitable positive constants C and μ er have,
$\|\vec{x}(t)\| \le c\|\overrightarrow{x_0}(t)\|e^{-\mu t}$ and $\lim_{t\to\infty} \vec{x}(t) = 0.$

(b) If Real $\boldsymbol{\lambda}_k\le 0$, (k = 1, 2,…,n) where the eigen values with $\boldsymbol{\lambda}_k = 0$ are distinct, then $\vec{x}(t)$ is bounded for $t \ge t_0$, Explicitly $\|\vec{x}(t)\| \le c\|\overrightarrow{x_0}\|$.

(c) If there exists an eigen values $\boldsymbol{\lambda}_k$ with real $\boldsymbol{\lambda}_k>0$ then each nbd of $\vec{x} = \vec{0}$, there are initial values such that the corresponding solutions we have,
$$\lim_{t\to\infty}\|\vec{x}(t)\| = +\infty.$$

In the case (a), then solution $\vec{x} = \vec{0}$ is asymptotically stable, in the case (b) $\vec{x} = \vec{0}$ is Liapunor stable and for the case (c), it is unstable.

Note: The solution the equation $\dot{\vec{x}} = A\vec{x}$ can be written in different way by using the concept of exponential matrix

$\vec{x}(t) = e^{At}\vec{c}$where $e^{At} = 1 + At + \frac{A^2t^2}{2!}$ ……………………(14.7)

The fundamental motion $\varphi(t)$ and inverse can be written as

$\varphi(t) = e^{At}$and $\varphi^{-1}(t) = e^{-At}$ ……………………………….(14.8)

II. Equations with coefficients which have a Limit:

Consider the equation

$\dot{\vec{x}} = A\vec{x} + B(t)\vec{x}$ …………………………(14.9)

with A, a non-singular constant nxn matrix B(t) a continuous nxn matrix. If$\lim_{t\to\infty}\|B(t)\| = 0$, then the solutions of (4.9)will tend to the solutions of $\dot{\vec{x}} = A\vec{x}$.

Theorem-14.2: Consider the equation$\dot{\vec{x}} = A\vec{x} + B(t)\vec{x}$ and suppose that

(a) The eigen values λ_k(k =1, 2,…,n)of A have real $\lambda_k \leq 0$, the eigen values corresponding to real λ_k are distinct

and (b) $\int_{t_0}^{\infty}\|B(t)\|dt$ is bounded then the solution of equation (4.9) are bounded and $\vec{x} = \vec{0}$ is Liapunor stable.

Theorem-14.3: Consider the equation$\dot{\vec{x}} = A\vec{x} + B(t)\vec{x}$, B(t) is continuous for $t \geq t_0$ with

(a) A is constant nxn matrix having eigen values λ_k(k = 1, 2,…,n)s.t real $\lambda_k < 0$

And (b)$\lim_{t\to\infty}\|B(t)\| = 0$, then solutions of equation(14.9) we have

$\lim_{t\to\infty}\|\vec{x}(t)\| = \vec{0}$ and $\vec{x} = \vec{0}$ is a asymptotically stable.

Theorem-14.3: Consider the equation$\dot{\vec{x}} = A\vec{x} + B(t)\vec{x}$, B(t) is continuous for $t \geq t_0$ and the property that $\lim_{t\to\infty}\|B(t)\| = 0$. If at least one eigen value of the matrix A has a $+ve$ real part, there exists in each nbd of $\vec{x} = \vec{0}$ solution $\vec{x}(t)$such that $\lim_{t\to\infty}\|\vec{x}(t)\| = +\infty$. The solution $\vec{x} = \vec{0}$is unstable.

III. Equations with Periodic Coefficients:

Consider the equation

$\dot{\vec{x}} = A(t)\vec{x}, t \in R$ …………………………………..(14.10)

with A(t) is continuous T-periodic nxn matrix i.e., A(t+T) = A(t).

Theorem-14.3:Floequet's Theorem:

Consider the equation $\dot{\vec{x}} = A(t)\vec{x}, t \in R$with A(t) is continuous T-periodic nxn matrix. Then each fundamental matrix $\varphi(t)$ of this equation can be written as the periodic nxn matrices in the form $\varphi(t) = \mathrm{P(t)e^{\beta t}}$ with P(t), T-periodic and β, a constant nxn matrix.

Proof: The fundamental matrix $\varphi(t)$ is compound of n-independent solutions and so $\varphi(t+T)$ is also fundamental matrix. If we put $\tau = t + T$. Then

$$\frac{d\vec{x}}{d\tau} = A(\tau - T)\vec{x} = A(\tau)\vec{x}.$$

$So\varphi(\tau)$ i.e., $\varphi(t+T)$ is fundamental matrix. The fundamental matrices $\varphi(t)$ and $\varphi(t+T)$ are linearly independent i.e., $\varphi(t+T) = \varphi(t)C$ where C is non-singular nxn matrix. There exists a constant matrix β such that $\mathrm{C} = e^{\beta t}$. We now proof that $\varphi(t)e^{-\beta t}$ is T-periodic. Let P(t)= $\varphi(t)e^{-\beta t}$ then

$$P(t+T) = \varphi(t+T)e^{-\beta(t+T)}$$

$$= \varphi(t)Ce^{-\beta t}e^{-\beta T}$$

$$= \varphi(t)Ie^{-\beta t}$$

$$= \varphi(t)e^{-\beta t}$$

$$= P(t)$$

Thus, P(t) and $\varphi(t)e^{-\beta t}$ is T-periodic.

14.6: Stability by Linearization:

The stability of linear solutions or periodic solutions can be studied by analysing the system, linearized in the nbd. of those special solutions. The justification of linearization method has been shown by Poincare′ and Liapunor. In this path we require Gronwall′s inequality(see article-1.3) given as follows-

Assume that for $t_0 \leq t \leq t_0 + a$, a being the positive constant, we have the estimate

$\varphi(t) = \delta_1 \int_{t_0}^{t} \Psi(s)\varphi(s)ds + \delta_2$ ……………………………(14.11)

In which $\varphi(t)\ and\ \Psi(t)$are continuous functions $\varphi(t) \geq 0\ and\ \Psi(t) \geq 0$ and $\delta_1\ and\ \delta_2$ are $+ve$ constant. Then we have for $t_0 \leq t \leq t_0 + a$

$\varphi(t) \leq \delta_2 e^{\delta_1 \int_{t_0}^{t} \Psi(s)ds}$ …………………………………………(14.12)

Asymptotic Stability of the Trivial Solution:

Theorem-14.6: Poincare′ Liapunor Theorem:

Consider the equation

$$\dot{\vec{x}} = A\vec{x} + B(t)\vec{x} + \vec{h}(\vec{x},t), \vec{x} \in D \subset R^n, t \in R, \vec{x}(t_0) = \overrightarrow{x_0} \quad (14.13)$$

Where A is a constant nxn matrix with eigen values which have all negative real points; B(t) continuous nxn matrix with the property $\lim_{t\to\infty}\|\vec{x}(t)\| = 0$. The vector function $\vec{h}(\vec{x},t)$ in continuous in $\vec{x}$ and t and Lipschitz continuous in $\vec{x}$ on nbd of $\vec{x} = \vec{0}$, Moreover we have $\lim_{t\to\infty}\frac{\|\vec{h}(\vec{x},t)\|}{\|\vec{x}(t)\|} = 0$ uniformly in t. Then there exists $+ve$ constant c, t_0, δ, μ such that $\|\vec{x}_0\| < \delta$ implies $\|\vec{x}(t)\| < c\|\vec{x}_0\|e^{(t-t_0)}, t \geq 0$. The solution $\vec{x} = \vec{0}$ asymptotically stable and the alteration is exponential in a $\delta - nbd$ of $\vec{x} = \vec{0}$.

Proof :

From theorem (14.1) we have estimate for the fundamental matrix $\varphi(t)$ of the equation $\dot{\varphi}(t) = A\varphi(t), \varphi(t_0) = I$ as $\varphi(t) = e^{(t-t_0)}$. As the eigen values of A has all non-zero real part, there exists $+ve$ constant c and such that

$$\|\vec{x}(t)\| < c\|\vec{x}_0\|e^{(t-t_0)}, t \geq t_0$$

It follows that the assumption on $\vec{h}$ and B that for $\delta > 0$ sufficiently smoll, there exists a constant b(δ) such that if $\|\vec{x}\| < \delta$ we have

$$\left\|\vec{h}(\vec{x},t)\right\| < \mathrm{b}(\delta)\|\vec{x}\|, t \geq t_0$$

and if t_0 is sufficiently large

$$\|B(t)\| < b(\delta), t \geq t_0$$

The existence and uniqueness theorem shows that, in the nbd of $\vec{x} = \vec{0}$, the solution of initial value problem (14.13)exits for $t_0 \leq t \leq t_1$ and therefore, this solution can be continued for all $t \geq t_0$.

Let, $\vec{x} = \varphi(t)\vec{z}$ and substitutes this equation (14.13) to obtain

$$\dot{\varphi}(t)\vec{z} + \varphi(t)\dot{\vec{z}} = A\varphi(t)\vec{z} + B(t)\varphi(t)\vec{z} + \vec{h}(\varphi(t)\vec{z},t)$$

$$i.e., \dot{\vec{z}} = \varphi^{-1}(t)B(t)\varphi(t)\vec{z} + \varphi^{-1}(t)\vec{h}(\varphi(t)\vec{z},t) as\ \dot{\varphi}(t) = A\varphi(t)$$

Integration of this expression and multiplication with $\varphi(t)$ produces for the solutions of equation (14.13), the integral equation

$\vec{x}(t) = \varphi(t)\vec{x}_0 + \int_{t_0}^{t} \varphi(t - s + t_0)[B(s)\vec{x}(s) + \vec{h}(\vec{x}(s), s)]ds$ ....(14.14)

where we have used the result

$$\varphi(t)\varphi^{-1}(s) = e^{A(t-t_0)}e^{-A(s-t_0)} = e^{A(t-s)} = \varphi(t - s + t_0)$$

Using the estimate for φ, B *and* $\vec{h}$we have for $t_0 \leq t \leq t_1$,

$$\|\vec{x}(t)\| \leq \|\varphi(t)\|\|\vec{x}_0\| + \int_{t_0}^{t} \|\varphi(t - s + t_0)\|\left[\|B(s)\|\|\vec{x}(s)\| + \left\|\vec{h}(\vec{x}(s), s)\right\|\right]ds$$

$\leq ce^{-\mu_0(t-t_0)}\|\vec{x}_0\| + \int_{t_0}^{t} ce^{-\mu(t-s)}2b\,\|\vec{x}(s)\|ds.$

So that

$e^{\mu_0(t-t_0)}\|\vec{x}(t)\| \leq c\|\vec{x}_0\| + \int_{t_0}^{t} ce^{\mu_0(s-t_0)}2b\,\|\vec{x}(s)\|ds.$

Putting $\varphi(t) = e^{\mu_0(t-t_0)}\|\vec{x}(t)\|, \Psi(t) = 2cb, \delta_1 = 1, \delta_2 = c\|\vec{x}_0\|$, we obtain from inequality (4.12)

$$e^{\mu_0(t-t_0)}\|\vec{x}(t)\| \leq c\|\vec{x}_0\|e^{2cb(t-t_0)}$$

$\|\vec{x}(t)\| \leq c\|\vec{x}_0\|e^{(2cb-\mu_0)(t-t_0)}$ ....(14.15)

If δ consequently b are small enough the $\mu = \mu_0 - 2cb$ is $+ve$ and we have the required estimate for $t_0 \leq t \leq t_1$. We choose $\|\vec{x}_0\|$ such that $c\|\vec{x}_0\| < \delta$. So the estimate (14.15) holds for all $t \geq t_0$.

Theorem-14.7:

Consider the equation

$\dot{\vec{x}} = A\vec{x} + B(t)\vec{x} + \vec{h}(\vec{x}, t), \vec{x} \in D \subset R^n, t \in R$....................................(14.16)

with A(t) is T-periodic, continuous matrix; the vector function $\vec{h}(\vec{x}, t)$ is continuous in $\vec{x}$and t and Lipschitz continuous in $\vec{x}$ for $t \in R$ in the nbd of $\vec{x} = \vec{0}$. Moreover we have $\lim_{t\to\infty} \frac{\|\vec{h}(\vec{x},t)\|}{\|\vec{x}(t)\|} = 0$ uniformly in t. If a real paths of the characteristic exponents of the characteristic exponents of linear periodic equation

$\dot{\vec{x}} = A(t)\vec{x}$ ... ... (14.17)

are negative, the solution $\vec{x} = \vec{0}$ of equation (14.16) is asymptotically stable and the attraction is the exponential in the $\delta - nbd$ of $\vec{x} = \vec{0}$.

Proof:

Let $\vec{x} = P(t)\vec{z}$ with P(t) a periodic matrix belonging to the fundamental matrix solution of equation (4.17). We find from (14.16)

$$\dot{P}(t)\vec{z} + P(t)\dot{\vec{z}} = A(t)P(t)\vec{z} + \vec{h}(P(t)\vec{z}, t)$$

$\dot{\vec{z}} = P^{-1}(AP - \dot{P})\vec{z} + P^{-1}\vec{h}(P\vec{z}, t)$ ……………………………….(14.18)

Putting P(t)= $\varphi(t)e^{-\beta t}$, where $\varphi(t)$ being fundamental matrix of (14.16) we have

$$\dot{P} = \dot{\varphi}e^{-\beta t} + \varphi e^{-\beta t}(-\beta)$$

$$= A\varphi e^{-\beta t} + \varphi e^{-\beta t}(-\beta)$$

$$AP - P\beta$$

Since $\dot{\varphi} = A\varphi$, hence (14.18) gives

$\dot{\vec{z}} = \beta\vec{z} + P^{-1}\vec{h}(P\vec{z}, t)$ ……………………………….(14.19)

The constant matrix β has only eigen values with $-ve$ real parts. The solution $\vec{z} = \vec{0}$ of solution (14.19) satisfies the requirements of the Poincare-Liapunor theorem from which the result follows.

Theorem 14.8:

Consider the equation in R^n is as follows

$\dot{\vec{x}} = A\vec{x} + B(t)\vec{x} + \vec{h}(\vec{x}, t), t \geq t_0$ ………………………….(14.20)

with A aconstant nxnmatrix having eigen values of which at least one has positive real part; B(t) is a continuous nxn matrix with the property $\lim_{t\to\infty}\|B(t)\|$. The vector function $\vec{h}(\vec{x}, t)$ is continuous in $\vec{x}$ and t;Lipschitz continuous in $\vec{x}$ in a nbd of $\vec{x} = \vec{0}$;if moreover we have $\lim_{\|\vec{x}\|\to 0}\frac{\|\vec{h}(\vec{x},t)\|}{\|\vec{x}\|} = 0$, uniformly in t.The trivail solution of equation (14.20) is unstable.

Proof:

Let δ be the non-singular constant nxn matrix. We put $\vec{x} = \delta$ in (14.20) to obtain

$\dot{\vec{y}} = \delta^{-1}A\delta\vec{y} + \delta^{-1}B(t)\delta\vec{y} + \delta^{-1}\vec{h}(\delta\vec{y}, t)$ ……………………………….(14.21)

The solution $\vec{x}(t)$ is real valued, $\vec{y}(t)$ was generally be complex function. In stability of the trivial solution of equation (14.21) implies stability of the trivial solution of 1(4.20). For simplicity we assume that δ can be chosen such that $\delta^{-1}A\delta$ is in diagonal form form i.e., the eigen values λ_i of the matrix A lie on the main diagonal of $\delta^{-1}A\delta$ and the other matrix elements are zero. We put Re(λ_i)≥ $\sigma > 0$, i = 1, 2, …, k and Re(λ_i) ≤ 0, i =

$k+1, k+2, \dots, n$. Let $R^2 = \sum_{i=1}^{k} |y_i|^2$ and $r^2 = \sum_{i=k+1}^{n} |y_i|^2$. Using (4.21) we compute the derivatives of R^2 and r^2; we shall use the result

$\frac{d}{dt}|y_i|^2 = \frac{d}{dt}(y_i\overline{y_i}) = \dot{y}_i\overline{y_i} + y_i\overline{y_i} = 2Re\lambda_i|y_i|^2 + (\delta^{-1}B(t)\delta\vec{y})_i\overline{y_i} + y_i(\delta^{-1}B(t)\delta\vec{y})_i + \left(\delta^{-1}\vec{h}(\delta\vec{y},t)\right)_i\overline{y_i} + y_i\left(\delta^{-1}\vec{h}(\delta\vec{y},t)\right)_i$.

Now we choose $\epsilon > 0, t_0, \delta$ and t. For $t \geq t_0$ and $\|\vec{y}\| < \delta$, we have

$|\delta^{-1}B(t)\delta\vec{y}|_i \leq \epsilon|\vec{y}|$,

$$\left|\delta^{-1}\vec{h}(\delta\vec{y},t)\right|_i \leq \epsilon|\vec{y}|$$

$$\therefore \frac{1}{2}\frac{d}{dt}(R^2 - r^2) \geq \sum_{i=1}^{k}(Re\lambda_i - \epsilon)|y_i|^2 - \sum_{i=k+1}^{n}(Re\lambda_i + \epsilon)|y_i|^2$$

We choose $0 < \epsilon < \frac{1}{2}\sigma$ and we have $\text{Re}\lambda_i - \epsilon \geq \sigma - \epsilon \geq \epsilon, i = 1, 2, \dots, k$ and $\text{Re}\lambda_i + \epsilon \leq \epsilon, for\ i = k+1, k+2, \dots, n.$

It follows that

$$\frac{1}{2}\frac{d}{dt}(R^2 - r^2) \geq \epsilon(R^2 - r^2), t \geq t_0, \|\vec{y}\| < \delta \quad \text{.........................(14.22)}$$

If we choose the initial values such that $(R^2 - r^2)_{t=t_0} = a > 0$ then we find from (4.22)$\|\vec{y}\|^2 \geq R^2 - r^2 \geq ae^{2\epsilon(t-t_0)}$. So this solution leaves the domain determined by $\|\vec{y}\| < \delta$; the trivial solution is unstable.

14.7: Rough-Hunwitz Criterion for Stability of Non-Linear Systems:

Consider the autonomous system

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \quad \text{..(14.23)}$$

In order to investigate the stability of the system near a chosen a critical point, we apply a sufficiently small disturbance to the system by choosing the $\vec{x}'s$ from their equilibrium values $\vec{x}_0(say)$. If, as time t increases in definitely, all the $\vec{x}'s$ return to their original equilibrium values with increasing t, the system is unstable.

Consider the small variation $\vec{\xi}$ of the equilibrium values $\vec{x}_0$for the $\vec{x}'s$ is given by $\vec{x} = \vec{x}_0 + \vec{\xi}$. Substituting in (14.23) and discarding term of order higher than the first in the $\vec{\xi}'s$, we get

$$\dot{\vec{\xi}} = A\vec{\xi} \qquad \text{..(14.24)}$$

$where$ A $= (a_{ij}) = \left(\frac{d\vec{f}}{d(x_1, x_2, \ldots, x_n)}\right)_{\vec{x}=\vec{x}_0}$ is a nxn constant matrix at the equilibrium state $\vec{x} = \vec{x}_0$. We assume that the matrix A is non-singular i.e., det A $\neq 0$.

The characteristic equation is

det $(A - \boldsymbol{\lambda I}) = 0$, when expanded, leads to an equation of the form

$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1} + a_n = 0.$ ....(14.25)

It was shown by Liapunor that if Re$\boldsymbol{\lambda} < 0$, the corresponding equilibrium state is stable; if at least of equation (14.25) has a $+ve$ real roots, the equilibrium isunstable.

We construct a set of n-determinants of the n-th degree equation (14.25) is as follows

$$\Delta_1 = |a_1|,\ \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ 0 & a_2 \end{vmatrix},\ \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ 0 & 0 & a_3 \end{vmatrix},\ \Delta_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & a_4 \end{vmatrix}$$

$$\text{and } \Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 \ldots\ldots\ldots 0 \\ a_2 & a_1 & a_0 & 0 \ldots\ldots\ldots 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} \ldots\ldots a_0 \\ 0 & 0 & 0 & 0 \ldots\ldots 0 \end{vmatrix}$$

The Rough-Hunwitz criterion states that Re $\boldsymbol{\lambda} < 0$, provided that all the coefficients that a_i(for i = 1, 2, ..., n) are $+ve$ and all the determinants Δ_i(for i $= 1, 2, \ldots, n$) are positive also.

Nothing that $\Delta_n = a_n\Delta_{n-1}$, it follows that for stability both $a_n > 0, \Delta_{n-1} > 0$.

14.8 Linear Stability Analysis:

So far we have relied on graphical methods to determine the stability of fixed points. Frequently one would like to have a more quantitative measure of stability, such as the rate of decay to a stable fixed point. This sort of information may be obtained by linearizing about a fixed point, as we now explain.

Let x^* be a fixed point, and let $\eta(t) = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we derive a differential equation for η. Differentiation yields

$$\dot{\eta} = \frac{\dot{v}}{\dot{w}}(x - x^*) = \dot{x}$$

since x^* is constant. Thus $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$. Now using Taylor's expansion we obtain

$$f(x^+ + \eta) = f(x^2) + \eta f^{'}(x^4) + O(\eta^2),$$

where $O(\eta^2)$ denotes quadratically small terms in η. Finally, note that $f(x^*) = 0$ since x^* is a fixed point. Hence

$$\eta = \eta f^{'}(x^*) + O(\eta^2).$$

Now if $f^{'}(x^*) \neq 0$, the $O(\eta^2)$ terms are negligible and we may write the approximation

$$\dot{\eta} = \eta f^{'}(x^*).$$

This is a linear equation in η, and is called the linearization about x^*. It shows that the perturbation $\eta(t)$ grows exponentially if $f^{'}(x^*) > 0$ and decays if $f^{'}(x^*) < 0$. If $f^{'}(x^*) = 0$, the $O(\eta^2)$ terms are not negligible and a nonlinear analysis is needed to determine stability, as discussed in Example below.

The upshot is that the slope $f^{'}(x^*)$ at the fixed point determines its stability. If you look back at the earlier examples, you'll see that the slope was always negative at a stable fixed point. The importance of the sign of $f^{'}(x^*)$ was clear from our graphical approach; the new feature is that now we have a measure of how stable a fixed point is-that's determined by the magnitude of $f^{'}(x^*)$. This magnitude plays the role of an exponential growth or decay rate. Its reciprocal $1/|f^{'}(x^4)|$ is a characteristic time scale; it determines the time required for $x(t)$ to vary significantly in the neighborhood of x^*.

EXAMPLE-1:
Using linear stability analysis, determine the stability of the fixed points for $\dot{x} = \sin x$.

Solution: The fixed points occur where $f(x) = \sin x = 0$. Thus $x^* = k\pi$, where k is an integer. Then

$$f^{'}(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd.} \end{cases}$$

Hence x^* is unstable if k is even and stable if k is odd.

EXAMPLE-2:
Classify the fixed points of the logistic equation, using linear stability analysis, and find the characteristic time scale in each case.

Solution: Here $f(N) = rN\left(1 - \frac{N}{K}\right)$, with fixed points $N^* = 0$ and $N^* = K$. Then $f^{'}(N) = r - \frac{2rN}{K}$ and so $f^{'}(0) = r$ and $f^{'}(K) = -r$. Hence $N^* = 0$ is unstable and $N^* = K$ is stable, as found earlier by graphical arguments. In either case, the characteristic time scale is $\frac{1}{|f^{'}(N^{*})|}=\frac{1}{r}$.

EXAMPLE-3:
What can be said about the stability of a fixed point when $f^{'}(x^{*})=0$?
Solution: Nothing can be said in general. The stability is best determined on a case-by-case basis, using graphical methods. Consider the following examples:
(a) $\dot{x}=-x^{3}$
(b) $x=x^{3}$
(c) $\dot{x}=x^{2}$
(d) $\dot{x}=0$

Each of these systems has a fixed point $x^{*}=0$ with $f^{'}(x^{*})=0$. However the stability is different in each case. Which shows that (a) is stable and (b) is unstable. Case (c) is a hybrid case we'll call half-stable, since the fixed point is attracting from the left and repelling from the right. We therefore indicate this type of fixed point by a half-filled circle. Case (d) is a whole line of fixed points; perturbations neither grow nor decay.

These examples may seem artificial, but we will see that they arise naturally in the context of bifurcations-more about that later.

14.9 Existence and Uniqueness

Our treatment of vector fields has been very informal. In particular, we have taken a cavalier attitude toward questions of existence and uniqueness of solutions tothe system $\dot{x}=f(x)$. That's in keeping with the "applied" spirit of this book. Nevertheless, we should be aware of what can go wrong in pathological cases.

EXAMPLE-4:
Show that the solution to $\dot{x}=x^{1/3}$ starting from $x_0=0$ is not unique.
Solution: The point $x=0$ is a fixed point, so one obvious solution is $x(t)=0$ for all t. The surprising fact is that there is another solution. To find it we separate variables and integrate:

$$\int x^{-1/3}dx=\int dt$$

so $\frac{3}{2}x^{2/3}=t+C$. Imposing the initial condition $x(0)=0$ yields $C=0$. Hence $x(t)=\left(\frac{2}{3}t\right)^{3/2}$ is also a solution!

When uniqueness fails, our geometric approach collapses because the phase point doesn't know how to move; if a phase point were started at the origin, would it stay there or would it move according to $x(t) = \left(\frac{2}{3}t\right)^{3/2}$? (Or as my friends in elementary school used to say when discussing the problem of the irresistible force and the immovable object, perhaps the phase point would explode!)

Actually, the situation in Example 2.5.1 is even worse than we've let on-there are infinitely many solutions starting from the same initial condition.

Existence and Uniqueness Theorem: Consider the initial value problem
$\dot{x} = f(x), \quad x(0) = x_0.$

Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis, and suppose that x_0 is a point in R. Then the initial value problem bas a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

For proofs of the existence and uniqueness theorem, see Borrelli and Coleman (1987), Lin and Segel (1988), or virtually any text on ordinary differential equations.
This theorem says that if $f(x)$ is smooth enough, then solutions exist and are unique. Even so, there's no guarantee that solutions exist forever, as shown by the

EXAMPLE-5:
Discuss the existence and uniqueness of solutions to the initial value problem $\dot{x} = 1 + x^2, x(0) = x_0$. Do solutions exist for all time?

Solution: Here $f(x) = 1 + x^2$. This function is continuous and has a continuous derivative for all x. Hence the theorem tells us that solutions exist and are unique for any initial condition x_0. But the theorem does not say that the solutions exist for all time; they are only guaranteed to exist in a (possibly very short) time interval around $t = 0$.

For example, consider the case where $x(0) = 0$. Then the problem can be solved analytically by separation of variables:

$$\int \frac{dx}{1 + x^2} = \int dt,$$

which yields

$$\tan^{-1} x = t + C$$

The initial condition $x(0) = 0$ implies $C = 0$. Hence $x(t) = \tan t$ is the solution. But notice that this solution exists only for $-\pi/2 < t < \pi/2$, because $x(t) \to \pm\infty$ as $t \to \pm\pi/2$. Outside of that time interval, there is no solution to the initial value problem for $x_0 = 0$.

The amazing thing about Example 2.5.2 is that the system has solutions that reach infinity in finite time. This phenomenon is called blow-up. As the name suggests, it is of physical relevance in models of combustion and other runaway processes, There are various ways to extend the existence and uniqueness theorem. One can allow f to depend on time t, or on several variables $x_1 \dots, x_i$. One of the most useful generalizations will be discussed later in Section 6.2.

From now on, we will not worry about issues of existence and uniqueness-our vector fields will typically be smooth enough to avoid trouble. If we happen to come across a more dangerous example, we'll deal with it then.

Exercises:

1. Discuss the stability of the equilibrium points of the systems with f(x) given by
   (a) $\begin{bmatrix} x_1 - x_1x_2 \\ x_2 - x_1^2 \end{bmatrix}$
   (b) $\begin{bmatrix} -4x_2 + 2x_1x_2 - 8 \\ 4x_2^2 - x_1^2 \end{bmatrix}$
   (c) $\begin{bmatrix} 2x_1 - 2x_1x_2 \\ 2x_2 - x_1^2 + x_2^2 \end{bmatrix}$
   (d) $\begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}$
   (e) $\begin{bmatrix} x_2 - x_1 \\ kx_1 - x_2 - x_1x_3 \\ x_1x_2 - x_3 \end{bmatrix}$.
   Hint: In 1 (e), the origin is a sink if $k < 1$ and a saddle if $k > 1$. It is a nonhyperbolic equilibrium point if $k = 1$.

2. Use the Liapunov function $V(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$ to show that the origin is an asymptotically stable equilibrium point of the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -x_2 - x_1x_2^2 + x_3^2 - x_1^3 \\ x_1 + x_3^3 - x_2^3 \\ -x_1x_3 - x_3x_1^2 - x_2x_3^2 - x_3^5 \end{bmatrix}.$$

   Show that the trajectories of the linearized system $\dot{x} = D\mathbf{f}(\mathbf{0})x$ for this problem lie on circles in planes parallel to the x_1, x_2 plane; hence, the origin is stable, but not asymptotically stable for the linearized system.

UNIT-15

Liapunov's criterion for stability. Stability of periodic solutions. Floquet's theorem.

15.1: Stability Analysis by Direct Method:

We now discusses a method of studying the stability of solution of the non-linear system without linearising it. For this, we first defined Liapunor functions.

Liapunor functions: Consider the equation

$\dot{\vec{x}} = \vec{f}(\vec{x}, t), t \geq t_0, \vec{x} \in D \subset R^n$ ....(15.1)

and assume that the trivial solution satisfied the equation $\vec{f}(\vec{0}, \vec{t}) = \vec{0}$, $t \geq t_0, \vec{0} \in D$. We introduce a function $\mathrm{V}(\vec{x}, t)$ which is defined and continuously differentiable in $[t_0, \infty]xD, D \subset R^n$. Moreover$\vec{x} = \vec{0}$ is an interior point of D and $\mathrm{V}(\vec{x}, t)$ does not depend explicitly on t and we write $\mathrm{V}(\vec{x})$.

The function $\mathrm{V}(\vec{x})$(with $\mathrm{V}(\vec{0})$=0) is called $+vely(-vely)$ definite in D in $\mathrm{V}(\vec{x})$>0(<0) for $\vec{x} \in D, \vec{x} \neq \vec{0}$. The function $\mathrm{V}(\vec{x})$(with $\mathrm{V}(\vec{0})$=0) is called $+vely(-vely)$ semi-definite in D if $\mathrm{V}(\vec{x}) \geq 0 (\leq 0)$ for $\vec{x} \in D, \vec{x} \neq \vec{0}$.

The function $\mathrm{V}(\vec{x}, t)$ is called $+vely(-vely)$ definite in D if there exists a function $w(\vec{x})$ such that $w(\vec{x})$ is defined and continuous in D, $w(\vec{0}) = 0, 0 < w(\vec{x}) \leq \mathrm{V}(\vec{x}, t)(\mathrm{V}(\vec{x}, t) \leq w(\vec{x}) < 0)$ for $\vec{x} \neq \vec{0}, t \geq t_0$.

To define semi-definite functions $\mathrm{V}(\vec{x}, t)$, we replace $<(>) \leq (\geq)$.

The function $\mathrm{V}(\vec{x})$ or $\mathrm{V}(\vec{x}, t)$ is called Liapunor function.

Some examples of Liapunor function in R^3 are as follows

$\mathrm{V}(\vec{x}) = x^2 + 2y^2 + 3z^2 + z^3 (+vely\ definite)$

$\mathrm{V}(\vec{x}) = x^2 + z^2 (+vely\ semi - definite)$

$\mathrm{V}(\vec{x}, t) = -x^2 \sin^2 t - y^2 - 4z^2 (-vely\ definite)$

where in all cases $D = \left\{\frac{x,y,z}{x^2} + y^2 + z^2 \leq 1\right\} and\ t \geq 0$.

The Orbital Derivatives:

The orbit derivative L_t of the function of the function $\mathrm{V}(\vec{x},t)$ in the direction of the vector field $\vec{x}$, where $\vec{x}$ is a solution of $\dot{\vec{x}} = \vec{f}(\vec{x},t)$ is defined by

$$L_t V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \vec{x}}\dot{\vec{x}}$$

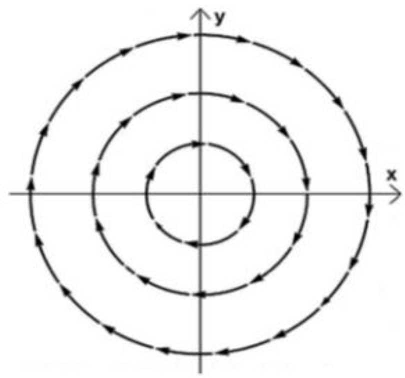
$$= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \vec{x}}\vec{f}(\vec{x},t)$$

$$= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(x_1,t) + \frac{\partial V}{\partial x_2} f_2(x_2,t) + \cdots + \frac{\partial V}{\partial x_n} f_n(x_n,t).$$

Theorem-15.1: First Theorem of Liapunov:

Consider the equation $\dot{\vec{x}} = \vec{f}(\vec{x},t)$ with $\vec{f}(\vec{0},t) = \vec{0}, \vec{x} \in D \subset R^n, t \geq t_0$. If a Liapunor function $\mathrm{V}(\vec{x},t)$ can be found, defined in a nbd of $\vec{x} = \vec{0}$ and positively definite for $t \geq t_0$ with orbital derivative negatively semi-definite, the solution $\vec{x} = \vec{0}$ is stable in the sense of Liapunor.

Proof:



In the nbd. of $\vec{x} = \vec{0}$ we have for the certain R>0 and $\|\vec{x}\| < R$, $\mathrm{V}(\vec{x},t) \geq \mathrm{w}(\vec{x})$ is defined and continuous in D, $\mathrm{w}(\vec{0}) = 0$ and L_t is the orbital derivative of V.

Consider the spherical shell B, given by $0 < r \leq \|\vec{x}\| < R$ and put $m = \min_{\vec{x}\in B} \mathrm{w}(\vec{x})$. Consider now a nbd δ of $\vec{x} = \vec{0}$ with the property that if $\vec{x} \in \delta, \mathrm{V}(\vec{x},t) < m$. Since $\mathrm{V}(\vec{x},t)$ is continuous and $+vely$ definitely with $\mathrm{V}(\vec{0},t) = 0$, such a nbd exists. The solution in δ at $t = t_0$, the solution can never enter B as we have for $t \geq t_0$.

$$\mathrm{V}(\vec{x}(t),t) - V(\vec{x}(t_0),t_0) = \int_{t_0}^{t} L_\tau\, V(\vec{x}(\tau),\tau)d\tau \leq 0.$$

In other words, the function$\mathrm{V}(\vec{x},t)$ can not increase alone a solution and this would be necessary to enter B as initially $V(\vec{x}(t_0),t_0)$<m.

We can repeat the argument for arbitrarily small R from which follows the stability.

Theorem-15.2: Second Theorem of Liapunov:

Consider the equation $\dot{\vec{x}} = \vec{f}(\vec{x}, t)$ with $\vec{f}(\vec{0}, t) = \vec{0}, \vec{x} \in D \subset R^n, t \geq t_0$. If a Liapunor function $V(\vec{x}, t)$can be found defined in a nbd of $\vec{x} = \vec{0}$ which for $t \geq t_0$ is $+vely$ definite in this nbd with $-vely$ definite orbital derivative, the solution $\vec{x} = \vec{0}$ is asymptotically stable.

Proof:

In the nbd$\vec{x} = \vec{0}$, we have $R > 0 \; and \; \|\vec{x}\| < R, V(\vec{x}, t) \geq w(\vec{x}) > 0, \vec{x} \neq \vec{0}, t \geq t_0$and $L_t V < 0$.

where $w(\vec{x})$ is defined and continuous in D, $w(\vec{0}) = 0$and L_t is a orbital derivative of V.

It follows from theorem-15.1 that $\vec{x} = \vec{0}$ is stable solution. Suppose that there is a solution $\vec{x}(t)$ and a constant a>0 such that $\|\vec{x}(t)\| \geq a \; for \; t \geq t_0$, when arbitrarily close to zero. The solution remains in the spherical shell B: $a \leq \|\vec{x}(t)\| \leq R, t \geq t_0$, we have $L_t V < -\mu, \mu > 0$.

$$\therefore V(\vec{x}(t), t) - V(\vec{x}(t_0), t_0) = \int_{t_0}^{t} L_\tau V(\vec{x}(\tau), \tau) d\tau$$

$$< -\mu(t - t_0)$$

On the other hand, we know that $V(\vec{x}, t)$ is $+vely$ definite, where as from the above $V(\vec{x}, t)$becomes $-ve$ after sometime. This is a contradiction. Hence the solution $\vec{x} = \vec{0}$ is asymptotically stable.

Note: A function $V(\vec{x}, t)$satisfying theorem-15.1 is called weak Liapunor function and that which satisfies theorem-15.2 is known as strong Liapunov function.

Theorem-15.3: Third Theorem of Liapunov:

Consider the equation $\dot{\vec{x}} = \vec{f}(\vec{x}, t)$ with $\vec{f}(\vec{0}, t) = \vec{0}, \vec{x} \in D \subset R^n, t \geq t_0$. If there exists a Liapunor function $V(\vec{x}, t)$in a nbd of $\vec{x} = \vec{0}$such that**:**

(a) $V(\vec{x}, t) \to 0 \; as \; \|\vec{x}\| \to 0$ uniformly in t.
(b) $L_t V$ is $+vely$ definite in the nbd of $\vec{x} = \vec{0}$.
(c) $From$ the certain nbd of $t = t_1 \geq t_0$, $V(\vec{x}, t)$takes $+ve$ values in each sufficiently small nbd of $\vec{x} = \vec{0}$.

Then the trivial solution $\vec{x} = \vec{0}$ is unstable.

Proof:

For certain $+ve$ constant aand b , we have with $\vec{x} \neq \vec{0}$ and $\|\vec{x}\| \leq a$ and $L_t V(\vec{x}, t) \geq w(\vec{x}) > 0\big(by\ (b)\big)$, where $w(\vec{x})$ defined and continuous in D, $w(\vec{0}) = 0$.

If possible suppose that $\vec{x} = \vec{0}$ is a stable solution. Then there exists an $\epsilon > 0$ with $\|\vec{x}_0\| \leq \epsilon$, we have$\|\vec{x}(t)\| \leq a\ for\ t \geq t_1$.

Using assumption (c), we can choose $\vec{x}_0$ such that $V(\overrightarrow{x_0}, t_1) > 0$. We find the solution $\vec{x}(t)$ which start in $\vec{x}_0$ at $t = t_1$.

$$\mathrm{V}(\vec{x}(t), t) - \mathrm{V}(\vec{x}_0, t_1) = \int_{t_1}^{t} L_\tau\, V(\vec{x}(\tau), \tau) d\tau > 0.$$

So, $V(\vec{x}, t)$ is non-decreasing. Consider now the set of points $\vec{x}$ with the property that $V(\vec{x}, t) \geq V(\overrightarrow{x_0}, t_1)$ and $\|\vec{x}\| \leq a$. This set is contained in the spherical shell B_1(say) given by $0 < r \leq \|\vec{x}\| \leq a$.

Let, $\mu = inf w(\vec{x}) > 0$

$\therefore \mathrm{V}(\vec{x}(t), t) - \mathrm{V}(\vec{x}_0, t_1) \geq \mu(t - t_1)$.

So, for $\|\vec{x}\| \leq a$, $V(\vec{x}, t)$ becomes arbitrarily large which is a contradiction. Hence the solution $\vec{x} = \vec{0}$ is unstable.

Note:

(1) Theorem (15.1)−(15.3) are also the true for autonomous equation $\dot{\vec{x}} = \vec{f}(\vec{x})$ and the fields are analogous.

(2) Let, the solution $\vec{x} = \vec{0}$ of the autonomous system $\dot{\vec{x}} = \vec{f}(\vec{x})$ be asymptotically stable. A set of points $\vec{x}_0$ with the property that for the solution of $\dot{\vec{x}} = \vec{f}(\vec{x})$,$\vec{x}(t) = \vec{x}_0$, we have
$\vec{x}(t) \to \vec{0}$ as $t \to \infty$ is called the domain of attraction of $\vec{x} = \vec{0}$.

Theorem-15.4:Floequet′s Theorem:

Consider the equation $\dot{\vec{x}} = A(t)\vec{x}, t \in R$with A(t) is continuous T-periodic nxn matrix. Then each fundamental matrix $\varphi(t)$ of this equation can be written as the periodic nxn matrices in the form $\varphi(t) = \mathrm{P(t)}e^{\beta t}$ with P(t), T-periodic and β, a constant nxn matrix.

Proof: The fundamental matrix $\varphi(t)$ is compound of n-independent solutions and so $\varphi(t + T)$ is also fundamental matrix. If we put $\tau = t + T$. Then

$$\frac{d\vec{x}}{d\tau} = A(\tau - T)\vec{x} = A(\tau)\vec{x}.$$

$So\varphi(\tau)$ i.e., $\varphi(t+T)$ is fundamental matrix. The fundamental matrices $\varphi(t)$ and $\varphi(t+T)$ are linearly independent i.e., $\varphi(t+T) = \varphi(t)C$ where C is non-singular nxn matrix. There exists a constant matrix β such that $C = e^{\beta t}$. We now proof that $\varphi(t)e^{-\beta t}$ is T-periodic. Let P(t)= $\varphi(t)e^{-\beta t}$ then

$$P(t+T) = \varphi(t+T)e^{-\beta(t+T)}$$

$$= \varphi(t)Ce^{-\beta t}e^{-\beta T}$$

$$= \varphi(t)Ie^{-\beta t}$$

$$= \varphi(t)e^{-\beta t}$$

$$= P(t)$$

Thus, P(t) and $\varphi(t)e^{-\beta t}$ is T-periodic.

15.2 Stability and Liapunov Functions:

In this section we discuss the stability of the equilibrium points of the nonlinear system

$$\dot{x} = \mathbf{f}(\mathbf{x}) \qquad (15.2)$$

The stability of any hyperbolic equilibrium point $\mathbf{x}_0$ of (15.2) is determined by the signs of the real parts of the eigenvalues λ_j of the matrix $D\mathbf{f}(\mathbf{x}_0)$. A hyperbolic equilibrium point x_0 is asymptotically stable iff $\mathrm{Re}(\lambda_j) < 0$ for $j = 1, \dots, n$; i.e., iff $\mathbf{x}_0$ is a sink. And a hyperbolic equilibrium point $\mathbf{x}_0$ is unstable iff it is either a source or a saddle. The stability of nonhyperbolic equilibrium points is typically more difficult to determine. A method, due to Liapunov, that is very useful for deciding the stability of nonhyperbolic equilibrium points is presented in this section.

Definition 1. Let ϕ_t denote the flow of the differential equation (15.2) defined for all $t \in \mathbf{R}$. An equilibrium point x_0 of (15.2) is stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\mathrm{x} \in N_\delta(\mathrm{x}_0)$ and $t \geq 0$ we have

$$\phi_t(\mathbf{x}) \in N_e(\mathbf{x}_0) =$$

The equilibrium point x_0 is unstable if it is not stable. And x_0 is asymptotically stable if it is stable and if there exists a $\delta > 0$ such that for all $\mathbf{x} \in N_\delta(\mathbf{x}_0)$ we have

$$\lim_{t\to\infty} \phi_t(\mathbf{x}) = \mathbf{x}_0$$

Note that the above limit being satisfied for all $\mathbf{x}$ in some neighborhood of x_0 does not imply that x_0 is stable. It can be seen from the phase portraits that a stable node or focus of a linear system in R^2 is an asymptotically stable equilibrium point; an unstable node or focus or a saddle of a linear system in $\mathbf{R}^2$ is an unstable equilibrium point; and a center of a linear system in $\mathbf{R}^2$ is a stable equilibrium point which is not asymptotically stable.

It follows from the Stable Manifold Theorem and the Hartman-Grobman Theorem that any sink of (15.2) is asymptotically stable and any source or saddle of (15.2) is unstable. Hence, any hyperbolic equilibrium point of (15.2) is either asymptotically stable or unstable. The corollary provides even more information concerning the local behavior of solutions near a sink:

Theorem 15.5. If x_0 is a sink of the nonlinear system (15.2) and $\mathrm{Re}(\lambda_j) < -\alpha < 0$ for all of the eigenvalues λ_j of the matrix $D\mathbf{f}(\mathbf{x}_0)$, then given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\mathbf{x} \in N_\delta(\mathbf{x}_0)$, the flow $\phi_t(\mathbf{x})$ of (1) satisfies

$$|\phi_t(\mathbf{x}) - \mathrm{x}_0| \leq \varepsilon e^{-\alpha t}$$

for all $t \geq 0$.
Since hyperbolic equilibrium points are either asymptotically stable or unstable, the only time that an equilibrium point x_0 of (15.2) can be stable but not asymptotically stable is when $D\mathrm{f}(\mathrm{x}_0)$ has a zero eigenvalue or a pair of complex-conjugate, pure-imaginary eigenvalues $\lambda = \pm ib$. It follows from the next theorem, proved in [H/S], that all other eigenvalues λ_j of $D\mathbf{f}(\mathbf{x}_0)$ must satisfy $\mathrm{Re}(\lambda_j) \leq 0$ if $\mathbf{x}_0$ is stable.

Theorem 15.6. If x_0 is a stable equilibrium point of (15.2), no eigenvalue of Df $(\mathbf{x}_0)$ has positive real part.

We see that stable equilibrium points which are not asymptotically stable can only occur at nonhyperbolic equilibrium points. But the question as to whether a nonhyperbolic equilibrium point is stable, asymptotically stable or unstable is a delicate question.

The following method, due to Liapunov (in his 1892 doctoral thesis), is very useful in answering this question.

Definition 2. If $\mathbf{f} \in C^1(E), V \in C^1(E)$ and ϕ_t is the flow of the differential equation (15.2), then for $\mathbf{x} \in E$ the derivative of the function $V(x)$ along the solution $\phi_t(\mathbf{x})$

$$\dot{V}(\mathbf{x}) = \frac{d}{dt} V(\phi_t(\mathbf{x}))\Big|_{\mathrm{t}=0} = DV(\mathbf{x})\mathrm{f}(\mathbf{x})$$

The last equality follows from the chain rule. If $\dot{V}(\mathbf{x})$ is negative in E then $V(\mathbf{x})$ decreases along the solution $\phi_t(\mathbf{x}_0)$ through $\mathrm{x}_0 \in E$ at $t = 0$. Furthermore, in R^2, if $\dot{V}(\mathrm{x}) \leq 0$ with equality only at $\mathrm{x} = 0$, then for small positive C, the family of curves $V(\mathbf{x}) = C$ constitutes a family of closed curves enclosing the origin and the trajectories of (1) cross these curves from their exterior to their interior with increasing t; i.e., the origin of (1) is asymptotically stable. A function $V\colon \mathbf{R}^n \to \mathbf{R}$ satisfying the hypotheses of the next theorem is called a Liapunov function.

Theorem 15.7. Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$. Suppose that $\mathrm{f} \in C^1(E)$ and that $\mathbf{f}(\mathbf{x}_0) = 0$. Suppose further that there exists a real valued function $V \in C^1(E)$ satisfying $V(\mathbf{x}_0) = 0$ and $V(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{x}_0$. Then (a) if $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in E, \mathrm{x}_0$ is stable; (b) if $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in E \sim \{x_0\}, x_0$ is asymptotically stable; (c) if $\dot{V}(\mathbf{x}) > 0$ for all $\mathbf{x} \in E \sim \{\mathrm{x}_0\}, \mathrm{x}_0$ is unstable.

Proof. Without loss of generality, we shall assume that the equilibrium point $\mathrm{x}_0 = 0$. (a) Choose $\varepsilon > 0$ sufficiently small that $\overline{N_c(0)} \subset E$ and let m_ε be the minimum of the continuous function $V(\mathbf{x})$ on the compact set

$$S_\varepsilon = \{\mathbf{x} \in \mathbf{R}^n \,||\mathbf{x}\, |= \varepsilon\}.$$

Then since $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$, it follows that $m_8 > 0$. Since $V(\mathbf{x})$ is continuous and $V(0) = 0$, it follows that there exists a $\delta > 0$ such that $|\mathbf{x}| < \delta$ implies that $V(\mathbf{x}) < m_\varepsilon$. Since $\dot{V}(\mathbf{x}) \leq 0$ for $\mathbf{x} \in E_,$, it follows that $V(\mathrm{x})$ is decreasing along trajectories of (1). Thus, if ϕ_t is the flow of the differential equation (1), it follows that for all $\mathrm{x}_0 \in N_\delta(0)$ and $t \geq 0$ we have

$$V\big(\phi_t(\mathrm{x}_0)\big) \leq V(\mathrm{x}_0) < m_\varepsilon.$$

Now suppose that for $|\mathbf{x}_0| < \delta$ there is a $t_1 > 0$ such that $\left|\phi_{\mathrm{t}_1}(\mathrm{x}_0)\right| = \varepsilon$; i.e., such that $\phi_{t_1}(\mathbf{x}_0) \in S_\varepsilon$. Then since m_ε is the minimum of $V(\mathbf{x})$ on S_c, this would imply that

$$V\left(\phi_{t_1}(\mathbf{x}_0)\right) \geq m_e$$

which contradicts the above inequality. Thus for $|\mathbf{x}_0| < \delta$ and $t \geq 0$ it follows that $|\phi_t(\mathrm{x}_0)| < \varepsilon_i$ i.e., 0 is a stable equilibrium point.

(b) Suppose that $V(x) < 0$ for all $x \in E$. Then $V(x)$ is strictly decreasing along trajectories of (1). Let ϕ_t be the flow of (1) and let $x_0 \in N_\delta(0)$, the neighborhood defined in part (a). Then, by part (a), if $|\mathrm{x}_0| < \delta$, $\phi_t(\mathrm{x}_0) \subset N_c(0)$ for all $t \geq 0$. Let $\{t_k\}$ be any sequence with $t_k \to \infty$. Then since $\overline{N_c(0)}$ is compact, there is a subsequence of $\left\{\phi_{t_k}(\mathrm{x}_0)\right\}$ that converges to a point in $\overline{N_\varepsilon(0)}$. But for any subsequence $\{t_n\}$ of $\{t_k\}$ such that $\left\{\phi_{t_n}(x_0)\right\}$ converges, we show below that the limit is zero. It then follows that $\phi_{t_k}(x_0) \to 0$ for any sequence $t_k \to \infty$ and therefore that $\phi_t(x_0) \to 0$ as $t \to \infty$; i.e., that 0 is asymptotically stable. It remains to show that if $\phi_{t_n}(\mathbf{x}_0) \to \mathbf{y}_0$, then $\mathbf{y}_0 = 0$. Since $V(\mathrm{x})$ is strictly decressing along trajectories of (1) and since $V\left(\phi_{t_x}(\mathbf{x}_0)\right) \to V(\mathbf{y}_0)$ by the continuity of V, it follows that

$$V\big(\phi_{\mathrm{t}}(\mathrm{x}_0)\big) > V(\mathrm{y}_0)$$

for all $t > 0$. But if $\mathbf{y}_0 \neq 0$, then for $s > 0$ we have $V\big(\phi_s(\mathbf{y}_0)\big) < V(\mathbf{y}_0)$ and, by continuity, it follows that for all $\mathbf{y}$ sufficiently close to $\mathbf{y}_0$ we have $V(\phi_s(\mathbf{y})) < V(\mathbf{y}_0)$ for $s > 0$. But then for $\mathbf{y} = \phi_{t_n}(\mathbf{x}_0)$ and n sufficiently large, we have

$$V\left(\phi_{s+\mathrm{t}_n}(\mathbf{x}_0)\right) < V(\mathbf{y}_0)$$

which contradicts the above inequality. Therefore $\mathbf{y}_0 = \mathbf{0}$ and it follows that 0 is asymptotically stable.

(c) Let M be the maximum of the continuous function $V(\mathbf{x})$ on the compact set $\overline{N_8(0)}$. Since $\dot{V}(\mathrm{x}) > 0, V(\mathrm{x})$ is strictly increasing along trajectories of (1). Thus, if ϕ_t is the flow of (1), then for any $\delta > 0$ and $\mathrm{x}_0 \in N_\delta(0) \sim \{0\}$ we have

$$V\big(\phi_t(\mathrm{x}_0)\big) > V(\mathrm{x}_0) > 0$$

for all $t > 0$. And since $\dot{V}(\mathbf{x})$ is positive definite, this last statement implies that

$$\inf_{t\geq 0} \dot{V}\big(\phi_t(\mathrm{x}_0)\big) = m > 0.$$

Thus,

$$V\big(\phi_t(\mathbf{x}_0)\big) - V(\mathbf{x}_0) \geq mt$$

for all $t \geq 0$. Therefore,

$$V\big(\phi_t(\mathbf{x}_0)\big) > mt > M$$

for t sufficiently large; i.e., $\phi_t(\mathbf{x}_0)$ lies outside the closed set $\overline{N_e(0)}$. Hence, 0 is unstable.

Remark. If $\dot{V}(\mathbf{x}) = 0$ for all $\mathbf{x} \in E$ then the trajectories of (1) lie on the surfaces in $\mathbf{R}^n$ (or curves in $\mathbf{R}^2$) defined by

$$V(\mathbf{x}) = c.$$

Example 15.1. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_2^3 \\ \dot{x}_2 &= x_1^3. \end{aligned}$$

The origin is a nonhyperbolic equilibrium point of this system and

$$V(\mathbf{x}) = x_1^4 + x_2^4$$

is a Liapunov function for this system. In fact

$$\dot{V}(\mathbf{x}) = 4x_1^3\dot{x}_1 + 4x_2^3\dot{x}_2 = 0.$$

Hence the solution curves lie on the closed curves

$$x_1^4 + x_2^4 = c^2$$

which encircle the origin. The origin is thus a stable equilibrium point of this system which is not asymptotically stable. Note that $Df(0) = 0$ for this example; i.e., $Df(0)$ has two zero eigenvalues.

Example 15.2. Consider the system

$$\begin{aligned} \dot{x}_1 &= -2x_2 + x_2x_3 \\ \dot{x}_2 &= x_1 - x_1x_3 \\ \dot{x}_3 &= x_1x_2. \end{aligned}$$

The origin is an equilibrium point for this system and

$$Df(0) = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $Df(0)$ has eigenvalues $\lambda_1 = 0, \lambda_{2,3} = \pm 2i$; i.e., $x = 0$ is a nonhyperbolic equilibrium point. So we use Liapunov's method. But how do we find a suitable Liapunov function? A function of the form

$$V(x) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2$$

with positive constants c_1, c_2 and c_3 is usually worth a try, at least when the system contains some linear terms. Computing $\dot{V}(\mathrm{x}) = DV(\mathbf{x})\mathrm{f(x)}$, we find

$$\frac{1}{2}\dot{V}(\mathbf{x}) = (c_1 - c_2 + c_3)x_1x_2x_3 + (-2c_1 + c_2)x_1x_2.$$

Hence if $c_2 = 2c_1$ and $c_3 = c_1 > 0$ we have $V(x) > 0$ for $\mathbf{x} \neq 0$ and $\dot{V}(\mathrm{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}^3$ and therefore by Theorem 3, $\mathrm{x} = 0$ is stable. Furthermore, choosing $c_1 = c_3 = 1$ and $c_2 = 2$, we see that the trajectories of this system lie on the ellipsoids $x_1^2 + 2x_2^2 + x_3^2 = c^2$.

We commented earlier that all sinks are asymptotically stable. However, as the next example shows, not all asymptotically stable equilibrium points are sinks. (Of course, a hyperbolic equilibrium point is asymptotically stable iff it is a sink.)

Example 15.3. Consider the following modification of the system in Example 2.

$$\begin{aligned}\dot{x}_1 &= -2x_2 + x_2x_3 - x_1^3\\ \dot{x}_2 &= x_1 - x_1x_3 - x_2^3\\ \dot{x}_3 &= x_1x_2 - x_3^3.\end{aligned}$$

The Liapunov function of Example 2,

$$V(\mathbf{x}) = x_1^2 + 2x_2^2 + x_3^2$$

satisfies $V(\mathbf{x}) > 0$ and

$$\dot{V}(\mathbf{x}) = -2(x_1^4 + 2x_2^4 + x_3^4) < 0$$

for $\mathbf{x} \neq \mathbf{0}$. Therefore, by Theorem 3 , the origin is asymptotically stable, but it is not a sink since the eigenvalues $\lambda_1 = 0, \lambda_{2,3} = \pm 2i$ do not have negative real part.
Example 4. Consider the second-order differential equation

$$\ddot{x} + q(x) = 0$$

where the continuous function $q(x)$ satisfies $xq(x) > 0$ for $x \neq 0$. This differential equation can be written as the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -q(x_1)\end{aligned}$$

where $x_1 = x$. The total energy of the system

$$V(\mathbf{x}) = \frac{x_2^2}{2} + \int_0^{x_1} q(s)ds$$

(which is the sum of the kinetic energy $\frac{1}{2}\dot{x}_1^2$ and the potential energy) serves as a Liapunov function for this system.

$$\dot{V}(\mathbf{x}) = q(x_1)x_2 + x_2[-q(x_1)] = 0.$$

The solution curves are given by $V(\mathbf{x}) = c$; i.e., the energy is constant on the solution curves or trajectories of this system; and the origin is a stable equilibrium point.

Example 15.4: Consider the system

$$\dot{x} = a(t)y + b(t)x(x^2 + y^2)$$

$$\dot{y} = -a(t)x + b(t)y(x^2 + y^2).$$

where the function a(t) and b(t) are continuous for $t \geq t_0$. Show that the trivial solution(0,0) is stable if b(t)≤ 0 and unstable if b(t)> 0 for $t \geq t_0$.

Solution:

Take V(x,y)= $(x^2 + y^2)$ as a Liapunor function.

Then, $L_t V = 2b(t)(x^2 + y^2)^2$.

Thus V is $+vely$ definite and $L_t V \leq 0$ if b(t)≤ 0 and $L_t V > 0$ if b(t)> 0. Hence, by theorem (15.1), the zero solution is stable if b(t)≤ 0 and by theorem (15.3), the zero solution is unstable if b(t)>0.

Example 15.5: Consider the equation for the non-linear oscillator with linear damping

$$\ddot{x} + \mu\dot{x} + x + ax^2 + bx^3 = 0$$

where $\mu, a \text{ and } b$ are constant and $\mu > 0$.

Solution:

We now introduce the energy of the non-linear oscillator without damping by

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + ax^3 + \frac{1}{4}bx^4.$$

We can find a nbd D of (0,0)dependent in size on a and b in which V is $+vely$ definite. Further more

$L_t V = \dot{x}\ddot{x} + x\dot{x} + ax^2\dot{x} + bx^3\dot{x} = -\mu\dot{x}^2$.

Application of theorem (15.1) shows that the solution (0,0) is Liapunor stable.

Example 15.6:

(i) Determined the stability of the zero solution of the system $\dot{x} = 2xy + x^3, \dot{y} = x^2 - y^5$.

Hint:$V = x^2 - 2y^2, L_t V = 2x^4 + 4y^6, unstable$.

(ii) Determine the stability of the trivial solution of $\dot{x} = xy^2 - \frac{1}{2}x^3, \dot{y} = -\frac{1}{2}y^3 + \frac{1}{5}x^2y$.

Hint: Asymptotically stable, V(x,y)= $x^2 + 2y^2$.

Example-15.7: Determine the stability of the zero solution of the system $\dot{x_1} = -x_2^3$; $\dot{x_2} = x_1^3$.

Solution:

The origin is equilibrium point of this system

$$\frac{dx_1}{dx_2} = \frac{\dot{x_1}}{\dot{x_2}} = -\frac{x_2^3}{x_1^3}.$$

$$\Rightarrow x_1^3 dx_1 + x_2^3 dx_2 = 0.$$

$$\Rightarrow x_1^4 + x_2^4 = c.$$

Hence V(x)=$x_1^4 + x_2^4$ is a Liapunor function. In fact V(x)= $4x_1^3\dot{x_1} + 4x_2^3\dot{x_2} = 0$.

Hence the solution curves $x_1^4 + x_2^4 = c^2$ which encircle of the origin. The origin thus stable equilibrium point of the system which is not asymptotically stable.

Then, $L_t V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \overrightarrow{x_1}}\dot{x_1} + \frac{\partial V}{\partial \overrightarrow{x_2}}\dot{x_2}$

$$= 4x_1^3\dot{x_1} + 4x_2^3\dot{x_2} + 4x_1^3\dot{x_1} + 4x_2^3\dot{x_2} = 8x_1^3(-x_2^3) + 8x_2^3(x_1^3) = 0.$$

Note that Df(0) = 0 for this example i.e., Df(0) has two zero eigen values.

Exercises:

1. Use the Liapunov function $V(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$ to show that the origin is an asymptotically stable equilibrium point of the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -x_2 - x_1x_2^2 + x_3^2 - x_1^3 \\ x_1 + x_3^3 - x_2^3 \\ -x_1x_3 - x_3x_1^2 - x_2x_3^2 - x_3^5 \end{bmatrix}.$$

Show that the trajectories of the linearized system $\dot{\text{x}} = D\text{f}(0)\text{x}$ for this problem lie on circles in planes parallel to the x_1, x_2 plane; hence, the origin is stable, but not asymptotically stable for the linearized system.

2. It was shown the origin is a center for the linear system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}.$$

The addition of nonlinear terms to the right-hand side of this linear system changes the stability of the origin. Use the Liapunov function $V(\mathbf{x}) = x_1^2 + x_2^2$ to establish the following results:

(a) The origin is an asymptotically stable equilibrium point of

$$\dot{\text{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -x_1^3 - x_1x_2^2 \\ -x_2^3 - x_2x_1^2 \end{bmatrix}.$$

(b) The origin is an unstable equilibrium point of

$$\dot{\text{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} x_1^3 + x_1x_2^2 \\ x_2^3 + x_2x_1^2 \end{bmatrix}.$$

(c) The origin is a stable equilibrium point which is not asymptotically stable for

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -x_1x_2 \\ x_1^2 \end{bmatrix}$$

What are the solution curves in this case?

5. Use appropriate Liapunov functions to determine the stability of the equilibrium points of the following systems:

(a) $\dot{x}_1 = -x_1 + x_2 + x_1x_2$

(a) $\dot{x}_2 = x_1 - x_2 - x_1^2 - x_2^3$

(b) $\dot{x}_1 = x_1 - 3x_2 + x_1^3\dot{x}_2 = -x_1 + x_2 - x_2^2$

UNIT-16

Solutions of nonlinear differential equations by perturbation method: Secular term. Nonlinear damping.

16.1:Perturbation Method: Secular Term:

One of the important method for solving non-linear differential equation is the perturbation method. The method is applicable of two equations in which small parameter is associated with the non-linear terms. In application, we develop the desired quantities in powers of small parameters multiplied by coefficients which one function of independent variables; we then determine the coefficients one by one usually by solving a sequence of linear equations.

Let us consider the differential of the type

$\ddot{x} + x + \mu f(x, \dot{x}, t) = 0$……………………………...(16.1)

where $f(x, \dot{x}, t)$ is a analytic function of $x, \dot{x}, t$ and periodic in t period of μ is a small parameter. The solution of (5.1) will be sought in the form of a series

$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots$ …………….(16.2)

By proceeding in this way we often a series difficulty in the form of series is called secular term i.e., the term which grows up indefinitely as $t \to \infty$ and thus destroy the convergence of the series solution.

As an example of the appearance of the secular term we consider the equation

$\ddot{x} + x + \mu x^3 = 0, 0 < \mu \ll 1$ ………………………..(16.3)

Let the conditions are

$x(0) = A, \dot{x}(0) = 0$ …………………………………..(16.4)

Substituting (16.2) in (16.3) and equating the coefficients of the successive powers of μ to zero. We get the following sequence of linear differential equation

$$\ddot{x}_0 + x_0 = 0$$

$\ddot{x}_1 + x_1 = -x_0^3$ ……………….(16.5)

and the condition (16.4) gives

$x_0(0) = A, x_i(0) = 0, \dot{x}_0(0) = 0, \dot{x}_i(0) = 0\ (for\ i = 1, 2, \ldots)$ ………..(16.6)

By virtue of this conditions, the first of equation (16.5) gives

$$x_0 = Acost$$

Hence the second equation of (16.5) becomes

$$\ddot{x}_1 + x_1 = -A^3 \cos^3 t$$

$$i.e., \ddot{x}_1 + x_1 = -\frac{1}{4}A^3(3cost + cos3t)$$

whose solution is

$$x_1 = -\frac{3}{8}A^3 tsint - \frac{1}{32}A^3(3cost - cos3t)$$

Here the first term is secular term which contains t. The appearance of secular term in this case may be explained as follows:

When $\mu = 0$, the solution is periodic with periodic 2π. However due to the presence of the non-linear term μx^3 in equation (16.3), the solution for $\mu \neq 0$ may not be periodic with the same period. Since the period of generating solution $x_0 = Acost$ is 2π, the subsequent term in (16.2) must take care of this variation in the period, this resulting appearance of the secular terms.

16.2: Application of Perturbation method for Obtaining Periodic Solutions of Some Non-Linear Differential Equations:

I. Autonomous System:

Consider the differential equation

$\frac{d^2x}{dt^2} + x = \mu f(x, \frac{dx}{dt})$ ……………………..(16.7)

where μ is a non-dimensional parameter assume to be small. We also assume that $f(x, \frac{dx}{dt})$ is a polynomial in x and $\frac{dx}{dt}$. When $\mu = 0$, the periodic solution (16.7) is readily obtained as a linear combination of sent and cost of period 2π. But $\mu \neq 0$, the frequency of periodic solution becomes unknown; accordingly, we replace the independent variable t by $\tau = \omega t$ where ω is a unknown frequency of the periodic solution. It is clear that the variable of x is of period 2π in τ. Putting $\tau = \omega t$ in (16.7) we get

$$\omega^2 \underset{..}{x} + x = \mu f(x, \omega \underset{.}{x})$$

where $\dot{x} = \frac{dx}{d\tau}$ and $\ddot{x} = \frac{d^2x}{d\tau^2}$.

Let the solution x(τ) of (16.8) develop in a power series w.r.t the small parameter μ, the coefficients in the series being periodic function in τ. So, we write

$x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) +$ …………………(16.9)

The $x_i(\tau), (i = 1,2, \dots)$ be periodic function of τ of period 2π. It addition, it is also necessary to develop the unknown quantity ω w.r.t μ i.e.,

$\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots$ …………….(16.10)

We now substitute (16.9) and (16.10) into (16.8) and the equate the coefficients of like of μ. Then we obtain the sequence of second order linear differential equations in $x_i(\tau)$ which also evolve the unknown quantities ω_i. Since only the periodic solution is under consideration and origin of τ may be chosen arbitrarily $\dot{x}(\tau) = 0$ at $\tau = 0$. This initial condition and the condition of periodicity serve to determine the unknown quantities in (16.9) and (16.10).

Explain the above method in details, we take the differential equation

$\frac{d^2x}{dt^2} + x = \mu x^3$ …………………………..(16.11)

Putting $\tau = \omega t$, we get

$\omega^2\ddot{x} + x = \mu x^3 = 0$ ……………………(16.12)

Substituting (16.9) and (16.10) into (16.12) we get

$(\omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots)^2(\ddot{x}_0 + \mu\ddot{x}_1 + \mu^2\ddot{x}_2 + \cdots) + (x_0 + \mu x_1 + \mu^2 x_2 + \cdots) +$
$\mu(x_0 + \mu x_1 + \mu^2 x_2 + \cdots)^3 = 0.$

Equating like power of μ, we get the following sequence of the linear differential equations:

$\mu^0: \omega_0^2\ddot{x}_0 + x_0 = 0$ …………………..(16.13)

$\mu^1: \omega_0^2\ddot{x}_1 + x_1 = -2\omega_0\omega_1\ddot{x}_0 - x_0^3.$ …………………..(16.14)

$\mu^2: \omega_0^2\ddot{x}_2 + x_2 = (-2\omega_0\omega_2 + \omega_1^2)\ddot{x}_0 - 2\omega_0\omega_1\ddot{x}_1 - 3x_0^2 x_1$ …………(16.15)

…….. ……… . ……… ………….

and so on.

The initial conditions are given by x(0)=A.$\dot{x}(0) = 0$.

Since, $x(\tau + 2\pi) = x(\tau)$, so $x_i(\tau + 2\pi) = x_i(\tau)$ ……………….(16.16)

and $x_0(0) = A, x_{i+1}(0) = 0\dot{x}_i(0) = 0, (i = 0, 1, 2, \dots)$ ………………………..(16.17)

Solving (16.13) we get by using the above condition

$$x_0 = A\cos\tau, \omega_0 = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots(16.18)$$

This zero order solution is called governing or generating solution.

Using (16.18), equation (16.14)leads to

$$\ddot{x}_1 + x_1 = \left(2\omega_1 - \tfrac{3}{4}A^2\right)A cos\tau - \tfrac{1}{4}A^3 cos3\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16.19)$$

The first term of right hand side is the secular term and so we must put $\omega_1 = \frac{3}{8}A^2$(if A = 0, then the generating solution and so all other solution are trivial).

By virtue of the condition (16.17), the solution of (16.19) is given by

$$x_1 = \tfrac{1}{32}A^3 - (cos\tau + cos3\tau) \quad \ldots\ldots\ldots\ldots\ldots\ldots(16.20)$$

In similar way we obtained

$$\omega_2 = -\tfrac{21}{256}A^4, x_2 = \tfrac{23}{1024}A^5 cos\tau - \tfrac{3}{128}A^5\cos^3\tau + \tfrac{1}{1024}A^5 cos5\tau \quad \ldots\ldots\ldots\ldots(16.21)$$

From (16.9), (16.18), (16.20) and (16.21), the solution of (16.11) upto an including term to the second order of μ is given by

$$x(t) = (A - \frac{1}{32}\mu A^3 + \frac{23}{1024}\mu^2 A^5 \cos\omega t + \left(\frac{1}{32}\mu A^3 - \frac{3}{128}\mu^2 A^5\right)\cos 3\omega t + \frac{1}{1024}\mu^2 A^5 cos5\omega t + \cdots \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(16.22)$$

Also by using (16.10) and the values of $\omega_i (i = 0, 1, 2, \ldots)$ obtained above, the frequency of ω is

$$\omega = 1 + \tfrac{3}{8}\mu A^2 - \tfrac{21}{256}\mu^2 A^4 + \cdots \quad \ldots\ldots\ldots\ldots\ldots(16.23)$$

Note: It is to be noted that the frequency of ω depends on the amplitude A of the oscillation.

II. Non-Autonomous System:

Consider the differential equation of the form

$$\frac{d^2x}{dt^2} + x = \mu f(x, \frac{dx}{dt}, t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(16.24)$$

where μ is a small parameter and $f(x, \frac{dx}{dt})$ is a periodic in t with period 2π. Toillustrate the perturbation method for obtaining periodic solution of equation (16.24) we rewrite it as follows

$$\frac{d^2x}{d\tau^2} + x = \mu f(x, \frac{dx}{d\tau}, \tau + \delta) \text{where } \tau = t - \delta. \quad \ldots\ldots\ldots\ldots..(16.25)$$

Contrary to the autonomous system, through the frequency of the desired periodic solution is known, the phase angle can not be assigned arbitrarily. Further an unknown phase angle δ must be introduced in respect of the initial conditions

$\dot{x}(\tau) = 0\ at\ \tau = 0.$ ……………………..(16.26)

In addition to x, it is also necessary to develop δ with respect to μ, i.e., we have

$x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \cdots$ ……………………(16.27)

$\delta = \delta_0 + \mu\delta_1 + \mu^2\delta_2 + \cdots$ ……………………………(16.28)

Then proceeding analogously as for the autonomous system, we determine the unknown quantities in the right hand side of (16.27) and (16.28).

As an example consider the Duffing′s equation with a term of dissipation

$\frac{d^2x}{dt^2} + x = \mu\left(-\alpha x - \beta x^3 - k\frac{dx}{dt} + F\cos t\right)$ ……………….(16.29)

which can be written as

$\ddot{x} + x = \mu[-\alpha x - \beta x^3 - k\dot{x} + F\cos(\tau + \delta)]$ ……………(16.30)

$where\ \tau = t - \delta, \dot{x} = \frac{dx}{d\tau}\ and\ \ddot{x} = \frac{d^2x}{d\tau^2}.$

Substituting (16.27) and (16.28) into (16.30) and the equating like powers of μ, we get,

$\mu^0: \ddot{x}_0 + x_0 = 0$ …………………..(16.31)

$\mu^1: \ddot{x}_1 + x_1 = -\alpha x_0 - \beta x_0^3 - k\dot{x}_0 + F\cos(\tau + \delta_0)$ …………………..(16.32)

$\mu^2: \ddot{x}_2 + x_2 = -\alpha x_1 - 3\beta x_0^2 x_1 - k\dot{x}_1 - F\delta_2\sin(\tau + \delta_1)$ …………(16.33)

…….. ……… . ……… ………….

and so on.

The unknown quantities in the above equations are to be determined by thecondition

$x_i(\tau + 2H) = x_i(\tau)(for\ i = 0, 1, 2, \dots)$ ……………………….(16.34)

and $\dot{x}_i(0) = 0. (for\ i = 0, 1, 2, \dots)$ …………………………(16.35)

Solving (16.31) with the condition $\dot{x}_0(0) = 0$, we get

$x_0(\tau) = A_0 \cos\tau$ ……………………………………………(16.36)

Substituting (16.36) into (16.32) we obtained

$$\ddot{x}_1 + x_1 = -\left(\alpha A_0 + \frac{3}{4}\beta A_0^3 - F\cos\delta_0\right)\cos\tau + (kA_0 - F\sin\delta_0)\sin\tau - \frac{1}{4}\beta A_0^3\cos 3\tau$$

……………………………………(16.37)

Periodicity condition for x_1 requires that there will be no regular term and therefore

$$\alpha A_0 + \frac{3}{4}\beta A_0^3 - Fcos\delta_0 = 0.$$

and $kA_0 - Fsin\,\delta_0 = 0.$ ……….(16.38)

The solution of (16.37) is then

$x_1(\tau) = A_1 \cos\tau + \frac{1}{32}\beta A_0^3 \cos 3\tau$ ……………………………(16.39)

The amplitude A_1 and phase angle $\delta_1$can be obtained by using the periodicity condition for $x_2(\tau)$ by summarizing the above results, the solution x(t) up to an including terms of first order in μ is found to be

$x(t) = (A_0 + \mu A_1)\cos(t - \delta_0 - \mu\delta_1) + \frac{1}{32}\mu\beta A_0^3(t - \delta_0 - \mu\delta_1)$ ………(16.40)

16.3 Use perturbation method to obtained solutions of period 2π and the amplitudefrequency relations up to including terms of order μ^2 for the following equation

$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0(Vander\ Pol\ equation)$, initial condition for problem (i) and (ii) are $x(0) = a, \dot{x}(0) = 0.$

Solution:

Let $\tau = \omega t$, then the given equation reduces to

$\omega^2\ddot{x} - \mu(1 - x^2)\omega\dot{x} + x = 0$……………….(1)

where $\dot{x} = \frac{dx}{d\tau}$ and $\ddot{x} = \frac{d^2x}{d\tau^2}$

Let, $x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \mu^3 x_3(\tau) + \cdots$

and $\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \mu^3\omega_3 + \cdots$

Substituting this in (1), we get

$(\omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots)^2(\ddot{x}_0 + \mu\ddot{x}_1 + \mu^2\ddot{x}_2 + \cdots) - \mu(\omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots)\{1 - (x_0 + \mu x_1 + \mu^2 x_2 + \cdots)\}(\dot{x_0} + \mu\dot{x_1} + \mu^2\dot{x_2} + \mu^3\dot{x_3} + \cdots) + (x_0 + \mu x_1 + \mu^2 x_2 + \cdots) = 0$

Equating like power of μ from both sides, we get

μ^0: $\omega_0^2\ddot{x}_0 + x_0 = 0$ …………………..(2)

μ^1: $\omega_0^2\ddot{x}_1 + x_1 = -2\omega_0\omega_1\ddot{x}_0 - \omega_0(1 - x_0^2)\dot{x}_0$ …………………..(3)

$\mu^2: \omega_0^2\ddot{x}_2 + x_2 = (-2\omega_0\omega_2 + \omega_1^2)\ddot{x}_0 - 2\omega_0\omega_1\dot{x}_1 + \omega_1(1 - x_0^2)\dot{x}_0 - 2\omega_0 x_0 x_1 \dot{x}_0 + \omega_0(1 - x_0^2)\dot{x}_1$……………………………..….(4)

$$\mu^3: \omega_0^2\ddot{x}_3 + x_3 = -2\omega_0\omega_2\ddot{x}_2 - (\omega_1^2 + 2\omega_0\omega_2)\ddot{x}_1 - 2(\omega_0\omega_3 + \omega_1\omega_2)\ddot{x}_0 + (1 - x_0^2)(\omega_0\dot{x}_2 + \omega_1\dot{x}_1 + \omega_2\dot{x}_0) - 2x_0x_1(\omega_0\dot{x}_1 + \omega_1\dot{x}_0) - \omega_0\dot{x}_0(2x_0x_2 + 2x_1^2)$$

…………………………………….(5)

The periodicity and initial conditions for $x_i(\tau)(i = 1, 2, ...)$ are

$x_i(\tau + 2\pi) = x_i(\tau), \dot{x}_i(0) = 0$ …………………………(6)

Solving (2) subject to the periodicity condition $x_0(\tau + 2\pi) = x_0(\tau) and$ the initial condition $\dot{x}_0(0) = 0$, we get $x_0(\tau) = \text{a}\cos\tau, \omega_0 = 1$, where the constant a is to be determine. The zero order solution $x_0(\tau)$ is a generating solution.

From (3), we have

$\ddot{x}_1 + x_1 = 2\tau\cos\tau + a\left(\frac{a^2}{4} - 1\right)\sin\tau + \frac{a^3}{4}\sin 3\tau$ ………………..(7)

Here the first and second term on the right hand side of (7) are secular term which destroyed the convergent of the required solution these we must have $a = 0, \pm 2\ and\ \omega_1 = 0$. Since $a_0 = 0$ provides amplitude, we must have $a = 2, \omega_1 = 0(a = -2$ provides no new information as it gives only a solution of opposite phase. Then the equation (7) reduces to

$$\ddot{x}_1 + x_1 = \frac{a^3}{4}\sin 3\tau = 2\sin 3\tau.$$

whose general solution is

$$x_1 = a_1\cos\tau + b_1\sin\tau - \frac{1}{4}\sin 3\tau$$

Since $\dot{x}_0(0) = 0$, we have $b_1 = \frac{3}{4}$.

Constant a_1 is to be determined by using equation (4) by putting $x_0(\tau) = 2\cos\tau$, $x_1 = a_1\cos\tau + \frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau, \omega_0 = 1, \omega_1 = 0$, then we have,

$\ddot{x}_2 + x_2 = \left(4\omega_2 + \frac{1}{4}\right)\cos\tau + 2a_1\sin\tau - \frac{3}{2}\cos 3\tau + 3a_1\sin 3\tau + \frac{5}{4}\sin 5\tau$ ……..(8)

Eliminating the secular term by putting

$\omega_2 = -\frac{1}{16}\ and\ a_1 = 0,$ we get

$x_1 = \frac{3}{4}\sin\tau - \frac{1}{4}sin3\tau$ and $\ddot{x}_2 + x_2 = -\frac{3}{2}\cos 3\tau + \frac{5}{4}\sin 5\tau$ whose solution is

$$x_2 = a_2 \cos\tau + b_2 \sin\tau + \frac{3}{16}\cos 3\tau - \frac{5}{96}\cos 5\tau\,.$$

Since $\dot{x}_2(0) = 0$, we have$b_2 = 0$ so that

$$x_2 = a_2 \cos\tau + \frac{3}{16}\cos 3\tau - \frac{5}{96}\cos 5\tau\,.$$

Constant a_2 is determined from the above equation. From (5) we have,

$$\ddot{x}_3 + x_3 = -2\omega_2\ddot{x}_1 - 2\omega_3\ddot{x}_0 + (1 - x_0^2)(\dot{x}_2 + \omega_2\dot{x}_0) - 2x_0x_1\dot{x}_1 - \dot{x}_0(2x_0x_1 + x_1^2)$$

$= \frac{1}{8}\left(-\frac{3}{4}\sin\tau - \frac{9}{4}\sin 3\tau\right) + 4\omega_3\cos\tau(1 + 4\cos^2\tau)\left(-a_2\sin\tau - \frac{9}{16}\sin 3\tau + \frac{25}{96}\sin 5\tau + \frac{1}{8}\sin\tau\right) + 2\sin\tau\{4\cos\tau\left(a_2\cos\tau + \frac{3}{10}\cos 3\tau - \frac{5}{96}\cos 5\tau\right) + \frac{1}{16}(3\sin\tau - \sin 3\tau)^2\}$

$$= 4\omega_3\cos\tau + \left(2a_2 + \frac{1}{4}\right)\sin\tau + \cdots$$

To determine the secular term all must put $\omega_3 = 0 \; and \; a_2 = -\frac{1}{8}$.

Hence, $x_2(\tau) = -\frac{1}{8}\cos\tau + \frac{3}{16}\cos 3\tau - \frac{5}{96}\cos 5\tau.$

Now we have,

$$x(\tau) = 2\cos\tau + \mu\left(\frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau\right) + \mu^2\left(-\frac{1}{8}\cos\tau + \frac{3}{16}\cos 3\tau - \frac{5}{96}\cos 5\tau\right) + \cdots$$

$and\; \omega = 1 - \frac{1}{16}\mu^2 + \cdots$ when $\tau = \omega t$.

16.4 FREE, DAMPED MOTION:

We now consider the effect of the resistance of the medium upon the mass on the spring. Still assuming that no external forces are present, this is then the case of free, damped motion. Hence with the damping coefficient $a > 0$ and $F(t) = 0$ for all t, the basic differential equation reduces to

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0. \qquad (16.41)$$

Dividing through by m and putting $k/m = \lambda^2$ and $a/m = 2b$ (for convenience) we have the differential equation (16.41) in the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \lambda^2 x = 0 \qquad (16.42)$$

Observe that since a is positive, b is also positive. The auxiliary equation is

$$r^2 + 2br + \lambda^2 = 0. \quad (16.43)$$

Using the quadratic formula we find that the roots of (16.43) are

$$\frac{-2b \pm \sqrt{4b^2 - 4\lambda^2}}{2} = -b \pm \sqrt{b^2 - \lambda^2}. \quad (16.44)$$

Three distinct cases occur, depending upon the nature of these roots, which in turn depends upon the sign of $b^2 - \lambda^2$.

Case 1. Damped, Oscillatory Motion. Here we consider the case in which $b < \lambda$, which implies that $b^2 - \lambda^2 < 0$. Then the roots (16.44) are the conjugate complex numbers $-b \pm \sqrt{\lambda^2 - b^2}\, i$ and the general solution of Equation (16.42) is thus

$$x = e^{-x^2}\left(c_1 \sin \sqrt{\lambda^2 - b^2}\, t + c_2 \cos \sqrt{\lambda^2 - b^2}\, t\right), \quad (16.45)$$

where c_1 and c_2 are arbitrary constants. We may write this in the alternative form

$$x = ce^{-bx} \cos\left(\sqrt{\lambda^2 - b^2}\, t + \phi\right), \quad (16.46)$$

where $c = \sqrt{c_1^2 + c_2^2} > 0$ and ϕ is determined by the equations

$$\frac{c_1}{\sqrt{c_1^2 + c_2^2}} = -\sin \phi_1.$$
$$\frac{c_2}{\sqrt{c_1^2 + c_2^2}} = \cos \phi.$$

The right member of Equation (16.46) consists of two factors,

$$ce^{-\mathrm{h}} \text{ and } \cos\left(\sqrt{\lambda^2 - b^2}\, t + \phi\right).$$

The factor ce^{-b} is called the damping factor, or time-zarying amplitude. Since $c > 0$, it is positive; and since $b > 0$, it tends to zero monotonically as $t \to \infty$. In other words, as time goes on this positive factor becomes smaller and smaller and eventually becomes negligible. The remaining factor, $\cos\left(\sqrt{\lambda^2 - b^2}\, t + \phi\right)$, is, of course, of a periodic, ascillatory character; indeed it represents a simple harmonic motion. The product of these two factors, which is precisely the right member of Equation (16.46), therefore represents an oscillatory motion in which the oscillations become successively smaller and smaller. The oscillations are said to be "damped out," and the motion is described as damped, oscillatory motion. Of course, the motion is no longer periodic, but the time interval between two successive (positive) maximum displacements is still referred to as the period. This is given by

$$\frac{2\pi}{\sqrt{\lambda^2 - b^2}}$$

The graph of such a motion is shown, in which the damping factor ce to and its negative are indicated by dashed curves.
The ratio of the amplitude at any time T to that at time

$$T - \frac{2\pi}{\sqrt{\lambda^2 - b^2}}$$

one period before T is the constant

$$\exp\left(-\frac{2\pi b}{\sqrt{\lambda^2 - b^2}}\right)$$

Thus the quantity $2\pi b/\sqrt{\lambda^2} - b^2$ is the decrease in the logarithm of the amplitude ce^{-t} over a time interval of one period. It is called the logarithmic decrement.

If we now return to the original notation of the differential equation (16.41), we see from Equation (16.46) that in terms of the original constants m, a, and k, the general solution of (16.46) is

$$x = ce^{-|\theta|2mi|t}\cos\left(\sqrt{\frac{k}{m} - \frac{a^2}{4m^2}}t + \phi\right). \quad (16.47)$$

Since $b < \lambda$ is equivalent to $a/2m < \sqrt{k/m}$, we can say that the general solution of (16.41) is given by (16.47) and that damped, oscillatory motion occurs when $a < 2\sqrt{\text{km}}$. The frequency of the oscillations

$$\cos\left(\sqrt{\frac{k}{m} - \frac{a^2}{4m^2}}t + \phi\right) \quad (16.48)$$

is

$$\frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{a^2}{4m^2}}.$$

If damping were not present, a would equal zero and the natural frequency of an undamped system would be $(1/2\pi)\sqrt{k/m}$. Thus the frequency of the oscillations (16.48) in the damped oscillatory motion (16.47) is less than the natural frequency of the corresponding undamped system.

Case 2. Critical Damping. This is the case in which $b = \lambda$, which implies that $b^2 - \lambda^2 = 0$. The roots (16.44) are thus both equal to the real negative number $-b$, and the general solution of Equation (16.42) is thus

$x = (c_1 + c_2 t)e^{-bt}$.   (16.49)

The motion is no longer oscillatory; the damping is just great enough to prevent oscillations. Any slight decrease in the amount of damping, however, will change the situation back to that of Case 1 and damped oscillatory motion will then occur. Case 2 then is a borderline case; the motion is said to be critically damped.

From Equation (16.49) we see that

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{c_1 + c_2 t}{e^{tr}} = 0.$$

Hence the mass tends to its equilibrium position as $t \to \infty$. Depending upon the initial conditions present, the following possibilities can occur in this motion:

1 The mass neither passes through its equilibrium position nor attains an extremum (maximum or minimum) displacement from equilibrium for $t > 0$. It simply approaches its equilibrium position monotonically as $t \to \infty$.

2 The mass does not pass through its equilibrium position for $t > 0$, but its displacement from equilibrium attains a single extremum for $t = T_1 > 0$. After this extreme displacement occurs, the mass tends to its equilibrium position monotonically as $t \to \infty$.

3 The mass passes through its equilibrium position once at $t = T_2 > 0$ and then attains an extreme displacement at $t = T_3 > T_2$, following which it tends to its equilibrium position monotonically as $t \to \infty$.

Case 3. Overcritical Damping. Finally, we consider here the case in which $b > \lambda_0$ which implies that $b^2 - \lambda^2 > 0$. Here the roots (16.41) are the distinct, real negative numbers

$$r_1 = -b + \sqrt{b^2 - \lambda^2}$$

and

$$r_2 = -b - \sqrt{b^2 - \lambda^2}$$

The general solution of (16.42) in this case is

$x = c_1 e^{r+1} + c_2 e^{r1e}$ (16.50)

The damping is now 50 great that no oscillations can occur. Further, we can no longer say that enery decrease in the amount of damping will result in oscillations, as we could in Case 2. The motion here is said to be overcritically damped (or simply overdamped). Equation (16.50) shows us at once that the displacement x approaches zero as $t \to \infty$. As in Case 2 this approach to zero is monotonic for 1 sufficiently large. Indeed, the three possible motions in Cases 2 and 3 are qualitatively the same. Thus the three motions illustrated can also serve to illustrate the three types of motion possible in Case 3.

Example16 .2:

A 32-lb weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to fest in its equilibrium position, thereby stretching the spring 2ft.The weight is then pulled down 6 in. below its equilibrium position and released at $t = 0$. No external forces are present; but the resistance of the medium in pounds is numerically equal to $4(dx/dt)$, where dx/dt is the instantaneous velocity in feet per second. Determine the resulting motion of the weight on the spring.

Formulation. This is a free, damped motion and Equation (16.41) applies. Since the $32 - 1\text{b}$ weight stretches the spring 2ft, Hooke's law, $F = ks$, gives $32 - k(2)$ and so $k = 16\text{lb/ft}$. Thus, since $m = w/g = \frac{32}{32} = 1$ (slug), and the damping constant $a = 4$. Equation (16.41) becomes

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 16x = 0 \qquad (16.51)$$

The initial conditions are

$$x(0) = \frac{1}{2}, \quad x'(0) = 0. \qquad (16.52)$$

Solution. The auxiliary equation of Equation (16.51) is

$$r^2 + 4r + 16 = 0 \qquad (16.53)$$

Its roots are the conjugate complex numbers $-2 \pm 2\sqrt{3}i$. Thus the general solution of (16.51) may be written

$$x = e^{-21}\left(c_1 \sin 2\sqrt{3}t + c_2 \cos 2\sqrt{3}t\right). \qquad (16.54)$$

where c_1 and c_2 are arbitrary constants. Differentiating (16.54) with respect to t we obtain

$$\frac{dx}{dt} = e^{-2t}\left[\left(-2c_1 - 2\sqrt{3}c_2\right)\sin 2\sqrt{3}t + \left(2\sqrt{3}c_1 - 2c_2\right)\cos 2\sqrt{3}t\right]. \qquad (16.55)$$

Applying the initial conditions (16.53) to Equations (16.54) and (16.55), we obtain

$$\begin{aligned} c_2 &= \frac{1}{2}, \\ 2\sqrt{3}c_1 - 2c_2 &= 0. \end{aligned}$$

Thus $c_1 = \sqrt{3}/6, c_2 = \frac{1}{2}$ and the solution is

$$x = e^{-2t}\left(\frac{\sqrt{3}}{6}\sin 2\sqrt{3}t + \frac{1}{2}\cos 2\sqrt{3}t\right) \quad (16.56)$$

Let us put this in the alternative form We have

$$\begin{aligned} \frac{\sqrt{3}}{6}\sin 2\sqrt{3}t + \frac{1}{2}\cos 2\sqrt{3}t &= \frac{\sqrt{3}}{3}\left[\frac{1}{2}\sin 2\sqrt{3}t + \frac{\sqrt{3}}{2}\cos 2\sqrt{3}t\right] \\ &= \frac{\sqrt{3}}{3}\cos\left(2\sqrt{3}t - \frac{\pi}{6}\right) \end{aligned}$$

Thus the solution (16.56) may be written

$$x = \frac{\sqrt{3}}{3}e^{-2t}\cos\left(2\sqrt{3}t - \frac{\pi}{6}\right). \quad (16.57)$$

Exercise:

(i) Use perturbation method to obtained solutions of period 2π and the amplitudefrequency relations up to including terms of order μ^2 for the following equations

(a) $\frac{d^2u}{dt^2} - \mu x\frac{dx}{dt} + x = 0.$

(b) $\left(1 + \mu\frac{dx}{dt}\right)\frac{d^2x}{dt^2} + x = 0.$

(c) $\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0 (Vander\ Pol\ equation)$, initial condition for problem (i) and (ii) are $x(0) = a, \dot{x}(0) = 0.$

UNIT-17

Solutions for the equations of motion of a simple pendulum, Duffing and Vanderpol oscillators.

17.1 Introduction:

At first almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This is so called small parameter assumptiongreatly restrict application of perturbation techniques. On Secondly, the determinationof small parameter seems to be a special art requiring special techniques. An appropriatechoice of small parameter leads to ideal result. However an unsuitable choice of smallparameter results badly. The Homotopy Perturbation method does not depend upona small parameter in the equation. This method, which is a combination of homotopyand perturbation techniques, provides us with a convenient way to obtain analytic orapproximate solution to a wide variety of problems arising in different field. So, this wasintroduced as a powerful tool to solve various kinds of non-linear problems.

17.2 Regular Perturbation Theory:

Very often, a mathematical problem cannot be solved exactly or, if the exact solution isavailable it exhibits such an intricate dependency in the parameters that it is hard to useas such. It may be the case however, that a parameter can be identified, say ϵ,such thatthe solution is available and reasonably simple for $\epsilon = 0$. Then one may wonder howthis solution is altered for non-zero but smallϵ. Perturbation theory gives a systematicanswer to this question.

Example-17.1: Consider an quadratic equation

$$x^2 - (3 + 2\epsilon)x + 2 + \epsilon = 0 \qquad (17.1)$$

when $\epsilon = 0$ then (17.1) reduce to

$$x^2 - 3x + 2 = 0 \Rightarrow (x - 2)(x - 1) = 0 \qquad (17.2)$$

whoee roots are $x = 1$ and 2. Equation (17.1) is called pertubed equation where as equation (17.2) is called un-perturbed or reduced equation.

Step1: In determining an approximate solution is to assume the form of the expansion. Let us assume that the roots have expansion in the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \qquad (17.3)$$

Here the first term x_0 is the zeroth-order term, the second term ϵx_1 is the first order term and the third term $\epsilon^2 x_2$ as the second order term.
Step2: Substitute equation (17.3) in equation (17.1)

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 - (3 + 2\epsilon)(x_0 + \epsilon x_1 + \cdots) + 2 + \epsilon = 0 \qquad (17.4)$$

Step3: Using binomial theorem to expand the first term

$$\begin{aligned}(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 &= x_0^2 + 2x_0(\epsilon x_1 + \epsilon^2 x_2 + \cdots) + (\epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 \\ &= x_0^2 + 2\epsilon x_0 x_1 + 2\epsilon^2 x_0 x_2 + \epsilon^2 x_1^2 + 2\epsilon^3 x_1 x_2 + \epsilon^4 x_2^2 + \cdots \\ &= x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0 x_2 + x_1^2) + \cdots \qquad (17.5)\end{aligned}$$

Similarly,

$$\begin{aligned}(3 + 2\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) &= 3x_0 + 3\epsilon x_1 + 3\epsilon^2 x_1 + 2\epsilon x_0 + 2\epsilon^2 x_1 + \cdots \\ &= 3x_0 + \epsilon(3x_1 + 2x_0) + \epsilon^2(3x_2 + 2x_1) + \cdots \end{aligned} \qquad (17.6)$$

Substitute equation (17.5)and(17.6) in equation (17.4)

$$x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0 x_2 + x_1^2) - \big(3x_0 + \epsilon(3x_1 + 2x_0) + \epsilon^2(3x_2 + 2x_1)\big) + 2 + \epsilon = 0$$

Collect the co-efficient of like powers of ϵ yields,

$$(x_0^2 - 3x_0 + 2) + \epsilon(2x_0 x_1 - 3x_1 - 2x_0 + 1) + \epsilon^2(2x_0 x_2 + x_1^2 - 3x_2 - 2x_1) + \cdots = 0$$
17.7)

Step4: Equating the co-efficient of each power of ϵ to Zero.

$$x_0^2 - 3x_0 + 2 = 0 \qquad (17.8)$$
$$2x_0 x_1 - 3x_1 - 2x_0 + 1 = 0 \qquad (17.9)$$
$$2x_0 x_2 + x_1^2 - 3x_2 - 2x_1 = 0 \qquad (17.10)$$

From equation (17.8), $x_0 = 1,2$, when $x_0 = 1$ equation (17.9) becomes

$$x_1 + 1 = 0 \Rightarrow x_1 = -1$$

When $x_0 = 1$ and $x_1 = -1$ equation (17.10) becomes

$$2x_2 + 1 - 3x_2 + 2 = 0$$
$$\Rightarrow x_2 - 3 = 0 \Rightarrow x_2 = 3$$

When $x_0 = 2$, equation (17.9) becomes

$$x_1 - 3 = 0 \Rightarrow x_1 = 3$$

equation (17.10) $\Rightarrow x_2 + 3 = 0 \Rightarrow x_2 = -3$

Step5: When $x_0 = 1, x_1 = -1$ and $x_2 = 3$

$Equ^n(3) \Rightarrow x = 1 - \epsilon + 3\epsilon^2 + \cdots$ (17.11)

When $x_0 = 2, x_1 = 3$ and $x_2 = -3$

$Eqc^5(3) \Rightarrow x = 2 + 3\epsilon - 3\epsilon^2$ (17.12)

∴ Hence $Equ^n(17.11)$ and (17.12) are the approximations for the two roots of (17.1). Now, to verify this approximation are correct, we compare with the exact solution.

$$\begin{aligned} x^2 - (3 + 2\epsilon)x + 2 + \epsilon &= 0 \\ \Rightarrow x &= \frac{1}{2}\left[3 + 2\epsilon \pm \sqrt{(3 + 2\epsilon)^2 - 4(2 + \epsilon)}\right. \\ \Rightarrow x &= \frac{1}{2}\left[3 + 2\epsilon \pm \sqrt{1 + 8\epsilon + 4\epsilon^2}\right] \quad (17.13) \end{aligned}$$

Using binomial theorem, we have

$$\begin{aligned} (1 + 8\epsilon + 4\epsilon^2)^{\frac{1}{2}} &= 1 + \frac{1}{2}(8\epsilon + 4\epsilon^2) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}(8\epsilon + 4\epsilon^2)^2 + \cdots \\ &= 1 + 4\epsilon + 2\epsilon^2 - \frac{1}{8}(64\epsilon^2 + \cdots) \\ &= 1 + 4\epsilon + 2\epsilon^2 - 8\epsilon^2 + \cdots \\ &= 1 + 4\epsilon - 6\epsilon^2 + \cdots \end{aligned}$$

Substitute this value in $Equ^n(17.13)$, we have

$$\begin{aligned} x &= \frac{1}{2}(3 + 2\epsilon + 1 + 4\epsilon - 6\epsilon^2 + \cdots) \\ &= 2 + 3\epsilon - 3\epsilon^2 + \cdots \\ x &= \frac{1}{2}(3 + 2\epsilon - 1 - 4\epsilon + 6\epsilon^2 + \cdots) \\ &= 1 - \epsilon + 3\epsilon^2 + \cdots \end{aligned}$$

Which are same as equation (17.11) and (17.12).

17.3 Singular Perturbation Theory:

It concern the study of problems featuring a parameter for which the solution of the problem at a limiting value of the parameter are different in character from the limit of the solution of the general problem. For regular perturbation problems, the solution of the general problem converge to the solution of the limit problem as the parameter approaches the limit value.

Example-17.2: Consider,

$$ex^2 + x + 1 = 0 \qquad (17.14)$$

Since equation (17.14) is a quadratic equation, it has two roots. For $\epsilon \longrightarrow 0$ Equation (17.14) reduce to

$$x + 1 = 0 \qquad (17.15)$$

Which is of first order. Thus x is discontinuous at $\epsilon = 0$. Such perturbation are called singular perturbation problem.

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \qquad (17.16)$$

Putting this value in Equation (17.14)

$$\begin{aligned} \epsilon(x_0 + \epsilon x_1 + \cdots) + x_0 + ex_1 + \cdots + 1 &= 0 \\ \Rightarrow \epsilon(x_0^2 + 2\epsilon x_0 x_1 + \cdots) + x_0 + \epsilon x_1 + \cdots + 1 &= 0 \\ \Rightarrow \epsilon x_0^2 + 2\epsilon^2 x_0 x_1 + \cdots + x_0 + \epsilon x_1 + \cdots + 1 &= 0 \\ \Rightarrow \epsilon(x_0^2 + x_1) + x_0 + 1 &= 0 \end{aligned}$$

Equating co-efficient of like power of ϵ gives

$$\begin{aligned} x_0 + 1 &= 0 \\ x_1 + x_0^2 &= 0 \end{aligned}$$

When $x_0 = -1, x_1 = -1$ So one of the root is

$$x = -1 - \epsilon + \cdots \qquad (17.17)$$

Thus as expected the above procedure yielded only one root. We investigate the exact solution i.e. ,

$$x = \frac{1}{2\epsilon}(-1 \pm \sqrt{1 - 4\epsilon}) \qquad (17.18)$$

Using binomial theorem we have

$$\begin{aligned} \sqrt{1 - 4\epsilon} &= 1 - 2\epsilon + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!} \times (-4\epsilon)^2 + \cdots \\ &= 1 - 2\epsilon - 2\epsilon^2 + \cdots \end{aligned} \qquad (17.19)$$

Substituting (17.19) in (17.18)

$$x = \frac{-1 + 1 - 2\epsilon - 2\epsilon^2 + \cdots}{2\epsilon} = -1 - \epsilon + \cdots \qquad (17.20)$$

$$x = \frac{-1 - 1 + 2\epsilon + 2\epsilon^2 + \cdots}{2\epsilon} = \frac{-1}{\epsilon} + 1 + \epsilon + \cdots \qquad (17.21)$$

Therefore, both of the roots go in powers of ϵ but one starts with ϵ^{-1}. Hence it is not surprising that the assumed expansion in (17.16) is failed to produce the root (17.21). consequently one can not determine the second root by a perturbation technique unless its form is known. In those cases, we recognize that, if the order of the equation is not to be reduced, the other tends to ∞ as $\epsilon \longrightarrow 0$ and hence, assume that the leading term has the form

$$x = \frac{y}{\epsilon^v} \qquad (17.22)$$

Where v must be greater than zero and needs to be determined in the course of analysis.
Substitute (17.22) in (17.14)

$$\epsilon^{1-2v} y^2 + \epsilon^v y + 1 + \cdots = 0$$

Since $v > 0$, th second term is much bigger than 1 . Hence the dominant part of (17.22) is

$$\epsilon^{1-2v} y^2 + \epsilon^v y = 0 \qquad (17.23)$$

which demands that power of ϵ be the same.

$$1 - 2v = -v \quad \Rightarrow v = 1$$

For $v = 1 \quad \Rightarrow y = 0$ or -1.
The first value $y = 0$, correspond to the first root $x = -1 - \epsilon$. For $y = -1$, it corresponds to second root. Thus it follows from (17.22)

$$x = \frac{-1}{\epsilon} + \cdots$$

To determine more terms in the expension of second root, we try

$$x = \frac{-1}{\epsilon} + x_0 + \cdots \qquad (17.24)$$

Substitute it in equation (17.14)

$$\begin{aligned} \Rightarrow \epsilon \left(\frac{-1}{\epsilon} + x_0 + \cdots\right)^2 - \frac{-1}{\epsilon} + x_0 + \cdots + 1 &= 0 \\ \Rightarrow \epsilon \left(\frac{-1^2}{\epsilon} + \frac{2x_0}{\epsilon} + x_0^2 + \cdots\right) - \frac{-1}{\epsilon} + x_0 + 1 + \cdots &= 0 \\ \Rightarrow -2x_0 + x_0 + 1 + O(\epsilon) &= 0 \end{aligned}$$

$\Rightarrow x_0 = 1$ and equation (17.24) becomes

$$x = -\frac{1}{\epsilon} + 1 + \cdots$$

Alternatively, once v has been determined. We view (17.22) as a transformation from x to y. Then putting $x = \frac{1}{\varepsilon}$ in (17.14) yields,

$$y^2 + y + \epsilon = 0 \qquad (17.25)$$

Which can be solved to determine both the roots because ϵ does not multiply the highest order.

17.4 Perturbation Theory For Differential Equation:

Example-17.3 : Consider,

$$\frac{d^2y}{d\tau^2} = -\epsilon\frac{dy}{d\tau} - 1, \quad y(0) = 0, \quad \frac{dy}{d\tau}(0) = 1 \qquad (17.26)$$

Let us assume the expansion

$$y(\tau) = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + O(\epsilon^3) \qquad (17.27)$$

Sutstitute Equation (17.27) in (17.26)

$$\frac{d^2y}{d\tau^2} + \epsilon\frac{dy}{d\tau} + 1 = 0$$

$$\frac{d^2}{d\tau^2}\left(y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + O(\epsilon^3)\right)$$

$$+\epsilon\frac{d}{d\tau}\left(y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + O(\epsilon^3)\right) + 1 = 0$$

$$\Rightarrow \frac{d^2y_0}{d\tau^2} + 1 + \epsilon\left(\frac{d^2y_1}{d\tau^2} + \frac{dy_0}{d\tau}\right) + \epsilon^2\left(\frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau}\right) + O(\epsilon^3) = 0$$

Equating the co-efficient of ϵ, it becomes

$$\Rightarrow \frac{d^2y_0}{d\tau^2} + 1 = 0, \quad y_0(0) = 0, \quad \frac{dy_0}{d\tau}(0) = 1$$

$$\Rightarrow \frac{d^2y_1}{d\tau^2} + \frac{dy_0}{d\tau} = 0, \quad y_1(0) = 0, \quad \frac{dy_1}{d\tau}(0) = 0 \qquad (17.28)$$

$$\Rightarrow \frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau} = 0, \quad y_1(0) = 0, \quad \frac{dy_1}{d\tau}(0) = 0$$

By solving the above equation we will get

$$y_0(\tau) = \tau - \frac{\tau^2}{2} \qquad (17.29)$$

$$y_1(\tau) = \frac{-\tau^2}{2} + \frac{\tau^3}{6} \qquad (17.30)$$

$$y_2(\tau) = \frac{\tau^3}{6} - \frac{\tau^4}{24} \qquad (17.31)$$

Putting these values in equation (17.27), we have the solution

$$y(\tau) = \tau - \frac{\tau^2}{2} + \epsilon\left(\frac{-\tau^2}{2} + \frac{\tau^3}{6}\right) + \epsilon^2\left(\frac{\tau^3}{6} - \frac{\tau^4}{24}\right) + O(\epsilon^3)$$

Example 17.4: We will consider the Lighthill equation

$$(x + \epsilon y)\frac{dy}{dx} + y = 0, \quad y(1) = 1 \qquad (17.32)$$

By the method, we can construct a homotopy which satisfies

$$(1-p)\left[\epsilon Y\frac{dY}{dx} - \epsilon y_0\frac{dy_0}{dx}\right] + p\left[(x+\epsilon y)\frac{dY}{dx} + Y\right] = 0, \quad p \in [0,1] \quad (17.33)$$

We can obtain a solution of (17.33) in the form

$$Y(x) = Y_0(x) + pY_1(x) + p^2Y_2(x) + \cdots \qquad (17.34)$$

Where $Y_i(x); i = 0,1,2,...$ are functions yet to be determined. By considering only first two terms of the above equation substitute equation (17.34) into equation (17.33)

$$(1-p)\left[\epsilon(Y_0 + pY_1)\left(\frac{dY_0}{dx} + \frac{dY_1}{dx}\right) - \epsilon y_0\frac{dy_0}{dx}\right]$$
$$+p\left[(x + \epsilon Y_0 + \epsilon p Y_1)\left(\frac{dY_0}{dx} + p\frac{dY_1}{dx}\right) + (Y_0 + pY_1)\right] = 0$$
$$\Rightarrow (1-p)\left[\epsilon Y_0\left(\frac{dY_0}{dx} + \frac{dY_1}{dx}\right) + \epsilon p Y_1\left(\frac{dY_0}{dx} + \frac{dY_1}{dx}\right) - \epsilon y_0\frac{dy_0}{dx}\right]$$
$$+p\left[(x + \epsilon Y_0 + \epsilon p Y_1)\left(\frac{dY_0}{dx} + p\frac{dY_1}{dx}\right) + (Y_0 + pY_1)\right] = 0$$
$$\Rightarrow \epsilon p Y_1\frac{dY_1}{dx} + (1-p)\left[\epsilon Y_0\frac{dY_0}{dx} - \epsilon y_0\frac{dy_0}{dx}\right]$$
$$+p\left[(x + \epsilon Y_0)\frac{dY_0}{dx} + Y_0\right] + \epsilon p^2 Y_1\left(\frac{dY_0}{dx} + p\frac{dY_1}{dx}\right) + p^2 Y_1 = 0$$

Now, we get

$$\epsilon Y_0 \frac{dY_0}{dx} - \epsilon y_0 \frac{dy_0}{dx} = 0 \qquad (17.35)$$

$$\epsilon Y_1 \frac{dY_1}{dx} + \left[(x + \epsilon Y_0)\frac{dY_0}{dx} + Y_0\right] = 0 \qquad (17.36)$$

The initial approximation $Y_0(x)$ or $y_0(x)$ can be freely chosen. Here I set

$$Y_0(x) = y_0(x) = -\frac{x}{\epsilon}, \quad Y_0(1) = -\frac{1}{\epsilon} \qquad (17.37)$$

So that, the residual of equation (17.32) at $x = 0$ vanishes. Then substitute equation (17.37) into equation (17.36),

$$\begin{aligned} \epsilon Y_1 \frac{dY_1}{dx} + \left[\left(x - \epsilon \frac{x}{\epsilon}\right)\frac{dY_0}{dx} - \frac{x}{\epsilon}\right] &= 0 \\ \Rightarrow \epsilon Y_1 \frac{dY_1}{dx} - \frac{x}{\epsilon} &= 0 \\ \Rightarrow \epsilon Y_1 \frac{dY_1}{dx} &= \frac{x}{\epsilon} \\ \Rightarrow \epsilon^2 Y_1 dY_1 &= x dx \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} &\Rightarrow \epsilon^2 \frac{Y_1^2}{2} = \frac{x^2}{2} + c \\ &\Rightarrow \epsilon^2 Y_1^2 = x^2 + 2c \\ &\Rightarrow Y_1 = \frac{\sqrt{x^2 + 2c}}{\epsilon} \\ &\Rightarrow \epsilon Y_1 = \sqrt{x^2 + 2c} \qquad (17.38) \end{aligned}$$

Putting the initial condition $Y_1(1) = 1 - Y_0 = 1 + \frac{1}{\varepsilon}$,

$$\begin{aligned} \Rightarrow \epsilon\left(1 + \frac{1}{\epsilon}\right) &= \sqrt{1 + 2c} \\ \Rightarrow 1 + \epsilon &= \sqrt{1 + 2c} \\ \Rightarrow 1 + \epsilon^2 + 2\epsilon &= 1 + 2c \\ \Rightarrow c &= \frac{\epsilon^2 + 2\epsilon}{2} \end{aligned}$$

Now, putting this value in equation (17.38) we get

$$Y_1 = \frac{1}{\epsilon}\sqrt{x^2 + 2\epsilon + \epsilon^2} \qquad (17.39)$$

Substitute this value in equ^n (17.34),

$$\Rightarrow Y(x) = Y_0(x) + Y_1(x) = \frac{1}{\epsilon}\left(-x + \sqrt{x^2 + 2\epsilon + \epsilon^2}\right) \qquad (17.40)$$

Which is the exact solution of $eqc°(17.32)$.

17.5Lindsted-Poincare′ Method:

Let us consider the differential equation

$\frac{d^2x}{dt^2} + \omega^2 x + \mu f(x) = 0$ ....(17.41)

which characterises a conservative system with unknown period T(say), i.e., unknown frequency $\Omega = \frac{2\pi}{T}$ which reduces to ω when $\mu \to 0$. In order to avoid dealing with unknown period, we select a variable $z(\tau)$ which has period 2π. If such a variable is selected, the system has a period 2π in that variable, but as we had to change the time scale as well, the frequency not Ω. In the new variables (i.e., in the variable $z(\tau)$ and $\tau = \Omega t$). The equation (17.41) becomes

$\Omega^2 \ddot{z} + \omega^2 z + \mu f(z) = 0$ ………………………(17.42)

where dot denotes the differentiation w.r.to τ. We put

$$z(\tau) = z_0(\tau) + \mu z_1(\tau) + \mu^2 z_2(\tau) + \cdots$$

And$\Omega^2 = \alpha_0 + \mu\alpha_1 + \mu^2\alpha_2 + \cdots$…………………(17.43)

It is seen that $\Omega^2 = \omega^2 = \alpha_0 \; as \; \mu \to 0$. Since in this case we have harmonic oscillation of period 2π in the new choice of time scale, we have also

$$f(z) = f(z_0 + \mu z_1 + \mu^2 z_2 + \cdots)$$

$$= f(z_0) + \mu z_1 f'(z_0) + \mu^2[z_2 f'(z_0) + \frac{z_1^2}{2!} f''(z_0)] + \cdots$$

Substituting (17.43) and (17.44) into (17.42) and then equating like power of μ from both sides we get, the sequence of linear differential equation

$$\omega^2 \ddot{z_0} + \omega^2 z_0 = 0$$

$$\omega^2 \ddot{z_1} + \omega^2 z_1 = -f(z_0) - \alpha_1 \ddot{z_0}$$

$$\omega^2 \ddot{z_2} + \omega^2 z_2 = -z_1 f'^{(z_0)} - \alpha_2 \ddot{z_0} - \alpha_1 \ddot{z_1}$$

$\omega^2 \ddot{z}_{n+1} + \omega^2 z_{n+1} = F(z_0, z_1, z_2, \dots, z_n) - \alpha_{n+1}\ddot{z_0} - \alpha_n \ddot{z_1} - \cdots - \alpha_1 \ddot{z_n}$.

where $F(z_0, z_1, z_2, \dots, z_n)$ is a polynomial in $z_0, z_1, z_2, \dots, z_n$.

The first equation (17.45) gives $z_0 = a\cos\tau$(in view of $z_0(0) = a \; and \; \dot{z}_0(0) = 0$), where a is arbitrary constant. Substituting this in the second equation of (17.45), we get

$\omega^2(\ddot{z}_1 + z_1) = -f(a\cos\tau) + \alpha_1 \, a\cos\tau$ ……………………..(17.46)

We develop the function $f(a\cos\tau)$ in the Fourier series containing only cosine term

$$i.e., f(f(a\cos\tau) = \sum_{n=0}^{\infty} f_n(a)\cos n\tau$$

$$= f_0(a) + f_1(a)\cos\tau + \sum_{n=2}^{\infty} f_n(a)\cos n\tau$$

So that the equation (17.46) becomes

$\omega^2(\ddot{z}_1 + z_1) = -f_0(a) + \{\alpha_1 a - f_1(a)\cos\tau\} - \sum_{n=2}^{\infty} f_n(a)\cos n\tau$ ……..(17.47)

Equating the secular term by putting $\alpha_1 = \frac{f_1(a)}{a}$, we get from (17.47), we have

$\ddot{z}_1 + z_1 = -\frac{1}{\omega^2} f_0(a) - \frac{1}{\omega^2}\sum_{n=2}^{\infty} f_n(a)\cos n\tau$ ……………….(17.48)

Whose solution is

$z_1 = A\cos\tau = -\frac{1}{\omega^2} f_0(a) + \frac{1}{\omega^2}\sum_{n=2}^{\infty} \frac{f_n(a)\cos n\tau}{n^2-1}$, where A is constant of integration.

To simplifying the solution we take $A = 0$, so that

$z_1(\tau) = A\cos\tau = -\frac{1}{\omega^2} f_0(a) + \frac{1}{\omega^2}\sum_{n=2}^{\infty} \frac{f_n(a)\cos n\tau}{n^2-1}$ ………………(17.49)

Replacing z_0 *and* z_1 into the third equation of (17.45) and eliminating the secular terms we obtained the value of α_2. The process is repeated and the sequence $z_0, z_1, z_2, \dots, z_n$ of successive approximation are obtained, but in each of term the singular term (resonance term) are eliminated, this results in another sequence $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ which is determine the frequency Ω^2. Thus we arrive the differential equation

$$\ddot{z}_{n+1} + z_{n+1} = \frac{1}{\omega^2}\left[b_0(a) + \sum_{n=2}^{\infty} b_n(a)\cos n\tau\right] + \frac{1}{\omega^2}[\alpha_{n+1}a + b_1(a)\cos\tau]$$

This elimination of the secular terms requires $\alpha_{n+1} = \frac{b_1(a)}{a}$ which results in the differential equations

$\ddot{z}_{n+1} + z_{n+1} = \frac{1}{\omega^2}[b_0(a) + \sum_{n=2}^{\infty} b_n(a)\cos n\tau]$, whose solution is

$z_{n+1} = \frac{1}{\omega^2}\left[b_0(a) - \sum_{n=2}^{\infty} \frac{b_n(a)\cos n\tau}{n^2-1}\right]$ …………………….(17.50)

As the example, consider the differential equation

$$\ddot{x} + x + \mu x^3 = 0$$

Taking $z_0(\tau) = a\cos\tau$ with $\omega^2 = 1, \alpha_0 = 1$, we have

$$\ddot{z_1} + z_1 = \left(\alpha_1 a - \frac{3}{4}a^3\right)\cos\tau - \frac{1}{4}a^3\cos 3\tau$$

Therefore, $z_1 = \frac{1}{32}a^3\cos 3\tau$

When we have, $\ddot{z_2} + z_2 = \left(\alpha_2 a - \frac{3}{128}a^5\right)\cos\tau - \frac{21}{128}a^5\cos 3\tau - \frac{3}{128}a^5\cos 5\tau$

Therefore, $z_2 = -\frac{21}{1024}a^5\cos 3\tau + \frac{1}{1024}a^5\cos 5\tau$.

Similar, from the above results we have the value of $x(t) and\ \Omega^2$.

17.6 Application of Lindsted-Poincare′ Method of Obtaining Periodic Solution in the Neighbourhood of the Centre of Non-Linear Conservative Systems:

Consider the conservative system governed by the differential equation

$\ddot{u} + f(u) = 0$ ……………….(17.51)

where f(u) is in general non-linear function of u. Let $u = u_0$ be a centre and as such the motion represented by (17.51) is oscillatory in the nbd of $u = u_0$ and put $x = u - u_0$ so that the equation (17.51) is transformed into

$\ddot{x} + f(x + u_0)$ ………..….(17.52)

Assuming f can be expanded in Taylor′s series we have,

$\ddot{x} + \sum_{n=1}^{\infty}\alpha_n x^n = 0$ ……………(17.53)

where $\alpha_n = \frac{1}{n!}f^n(u_0)$ and $f^n(u_0)$ denotes the n-th derivative w.r.to arguments for the centre and $f(u_0) = 0,\ f^{'(u_0)} = \alpha_1 > 0$.

For small but finite amplitude motion, we introduce a small dimension parameter μ which is of the order of the motion. Hence, we assume that the solution of (17.53) can be represented by the expansion of the form

$x(t;\mu) = \mu x_1(t) + \mu^2 x_2(t) + \mu^3 x_3(t) + \cdots$ ………..(17.54)

Substituting (17.54) in (17.53) and then equating like power of μ, we obtain

$$\ddot{x_1} + \omega_0^2 x_1 = 0$$

$$\ddot{x_2} + \omega_0^2 x_2 = -\alpha_2 x_1^2$$

$$\ddot{x_3} + \omega_0^2 x_3 = -2\alpha_2 x_1 x_2 + \alpha_3 x_1^2$$

$$\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots$$

where $\omega_0^2 = \alpha_1$

Let us now introduce a $\mu - independent$variable τ by $\tau = \omega t$, where ω is a unspecified function of μ. Then assuming expansion of the form

$$\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots$$

$$\text{and } x(\tau;\mu) = \mu x_1(\tau) + \mu^2 x_2(\tau) + \mu^3 x_3(\tau) + \cdots$$

where $x_1, x_2, x_3, \ldots$ are independent of μ, the equation (17.53) gives

$$(\omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots)^2 \frac{d^2}{d\tau^2}(\mu x_1 + \mu^2 x_2 + \mu^3 x_3 + \cdots)$$
$$+ \sum_{n=1}^{\infty} \alpha_n \left((\mu x_1 + \mu^2 x_2 + \mu^3 x_3 + \cdots)\right)^n = 0.$$

Equating like powers of μ and taking $\omega_0^2 = \alpha_1$, we get

$$\frac{d^2 x_1}{d\tau^2} + x_1 = 0,$$

$$\omega_0^2\left(\frac{d^2 x_1}{d\tau^2} + x_1\right) = -2\omega_0\omega_1 \frac{d^2 x_1}{d\tau^2} - \alpha_2 x_1^2 .$$

17.7 Problem: Determine a two term of expansion for the frequency-amplitude relationship for the system generated by the equation

$$\ddot{u} + \omega_0^2 u(1 + u^2)^{-1} = 0$$

Solution:

Here $u = 0$ is the centre. Putting $\tau = \omega t$, the given system reduces to

$\omega^2 \frac{d^2 u}{d\tau^2} + \omega_0^2(u - u^3 + u^5 + \cdots) = 0$ ………………..(1)

Assuming expansion of the form

$$\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots$$

$$\text{and } x(\tau;\mu) = \mu u_1(\tau) + \mu^2 u_2(\tau) + \mu^3 u_3(\tau) + \cdots$$

………………………………………..(2)

We have from (1),

$$(\omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots)^2 \frac{d^2}{d\tau^2}(\mu u_1 + \mu^2 u_2 + \mu^3 u_3 + \cdots) + \omega_0^2[(\mu u_1 + \mu^2 u_2 + \mu^3 u_3 + \cdots) - (\mu u_1 + \mu^2 u_2 + \mu^3 u_3 + \cdots)^3 + (\mu u_1 + \mu^2 u_2 + \mu^3 u_3 + \cdots)^5 - \cdots] = 0.$$

Equating like power of μ from both sides we obtained

μ: $\omega_0^2 \frac{d^2u_1}{d\tau^2} + \omega_0^2 u_1 = 0$ ……………………………………………………….(3)

μ^2: $\omega_0^2 \frac{d^2u_2}{d\tau^2} + \omega_0^2 u_2 = -2\omega_0\omega_1 \frac{d^2u_1}{d\tau^2}$……………………..……………..(4)

μ^2: $\omega_0^2 \frac{d^2u_3}{d\tau^2} + \omega_0^2 u_3 = -\omega_1^2 \frac{d^2u_1}{d\tau^2} - 2\omega_0\omega_1 \frac{d^2u_2}{d\tau^2} + \omega_0^2 u_1^3 - 2\omega_0\omega_2 \frac{d^2u_4}{d\tau^2}$ ………(5)

Solution of equation (3) is

$$u_1 = a\cos(\tau + \epsilon)$$

where a and ϵ are arbitrary constant. Substituting this in (4) we get,

$$\frac{d^2u_2}{d\tau^2} + u_2 = -\frac{2\omega_0}{\omega_1} a\cos(\tau + \epsilon)$$

To avoid the secular term we must put$\omega_1 = 0$ and the solution of the above equation becomes

$$u_2 = b\cos(\tau + \epsilon)$$

Thus the equation (5) reduces to

$$\frac{d^2u_3}{d\tau^2} + u_3 = u_1^3 - 2\frac{\omega_2}{\omega_0}\frac{d^2u_1}{d\tau^2}$$

$$= \frac{a^3}{4}[\cos 3(\tau + \epsilon) + 3(\tau + \epsilon)] + 2\frac{\omega_2}{\omega_0}\ b\cos(\tau + \epsilon)$$

$$= \frac{1}{4}a^3\cos 3(\tau + \epsilon) + (\frac{3}{4}a^3 + 2a\frac{\omega_2}{\omega_0})\cos(\tau + \epsilon)$$

Eliminating the secular term gives

$$\omega_2 = -\frac{3}{8}a^2\omega_0.$$

Thus, $\omega = \omega_0 - \frac{3}{8}\omega_0 a^2\mu^2$

Exercises:

Determine a two term of expansion for the frequency-amplitude relationship for the system generated by the equations

(i) $\ddot{u} + \omega_0^2 u(1 + u^2)^{-1} = 0$

Ans: $\omega = \omega_0 - \frac{3}{8}\omega_0 a^2 \mu^2$

(ii) $\dot{u} + \omega_0^2 u + \alpha u^5 = 0$

Ans: $\omega = \omega_0 + \frac{5}{16}\omega_0^{-1}\alpha a^4 \mu^2$

(iii) $\ddot{u} - u + u^3 = 0$

Ans: $\omega^2 = 2 - 3a^2\mu^2$

(iv) $\dot{u} + \omega_0^2 u + \alpha u^2 \ddot{u} = 0$

Ans: $\omega = \omega_0 - \frac{1}{4}\omega_0 \alpha a^2 \mu^2$

UNIT-18

Bifurcation Theory: Origin of Bifurcation, Bifurcation Value, Normalisation, Resonance, Stability of a fixed point.

18.1: Introduction:

Bifurcation means a structural change in the orbit of a system. The bifurcation of a system had been first reported by the French mathematician Henri Poincaré in his work. The study of bifurcation is concerned with how the structural change occurs when the parameter(s) are changing. The structural change and the transition behaviour of a system are the central part of dynamical evolution. The point at which bifurcation occurs is known as the bifurcation point. The behaviour of fixed point and the nature of trajectories may change dramatically at bifurcation points. The characters of attractor and repellor are altered, in general when bifurcation occurs. The diagram of the parameter values versus the fixed points of the system is known as the bifurcation diagram. This chapter deals with important bifurcations of one and two-dimensional systems, their mathematical theories, and some physical applications.

The dynamics of a continuous system $\dot{x} = f(x, \mu)$ depends on the parameter$\mu \epsilon R$. It is often found that μ crosses a critical value, the properties of dynamicalevolution, e.g., its stability, fixed points, periodicity etc. may change. Moreover, acompletely new orbit may be created. Basically, a structurally unstable system istermed as bifurcation. The bifurcation diagram is very useful in understanding thedynamical behaviour of a system. Bifurcations associated with a single parameter arecalled codimension-1 bifurcations. On the other hand, bifurcations connected withtwo parameters are known as codiemension-2 bifurcations. These bifurcations givemany interesting dynamics and have a wide range of applications in biological andphysical sciences. Various bifurcations and their theories are the integral part ofnonlinear systems. We discuss some important bifurcations in one- and twodimensionalsystems in the following sections.

In preceding chapters, we have considered equations which contained parameters. For different value of this parameters, the behaviour of the solutions can be qualitatively different. Consider for instance the Van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \mu > 0.$$

If$\mu = 0$, then all solutions are periodic and the origin of the phase-plane is a centre. If $0 < \mu < 1$, the origin is an unstable focus and there exists asymptotically stable periodic solution corresponding with a limit cycle around the origin.

In this chapter we shall discus change of nature of critical points and branching of solutions when a parameter passes a certain value; all this is called bifurcation theory.

Example -18.1:

Consider the equation

$\dot{x} = \mu x - x^2$ ……………………….(18.1)

The trivial solution $x = 0$ is an equilibrium solution of (18.1). Another equilibrium solution is $x = \mu$. This solution coincide if $\mu = 0$; at the value $\mu = 0$ both $+ve\ and - ve$ values of μ, a non-trivial solution branches off $x = 0$. In passing the value $\mu = 0$ an exchange of stability of equilibrium solution $x = 0\ and\ x = \mu$ takes place. This is illustrated in the bifurcation diagram of (18.1) which gives the equilibrium solution as a function of the bifurcation parameter μ.

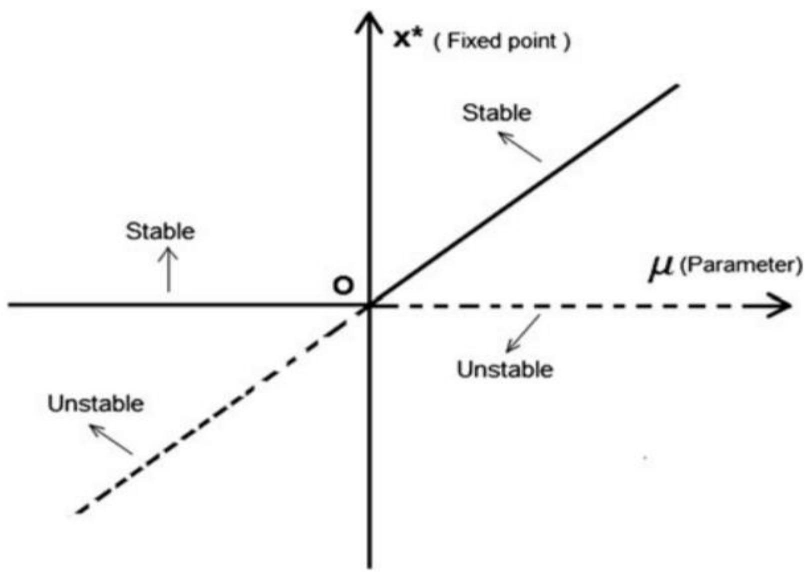


Fig.-18.1 (Bifurcation Diagram)

In the above example we can predict the possibilities of the existence of a branching or bifurcation point by the implicit function theorem, for the solution of x of the equation $F(\mu, x) = 0$ exists and unique if $\frac{\partial F}{\partial x} \neq 0$.

Regarding critical points of differential equations we consider the equations or system of equations like

$F(\mu, x) = 0$ …………………………..(18.2)

with $\mu \in R^m, x \in R^n$.

We now consider whether a solution (6.1) can bifurcate at certain values of the parameters $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$. By translation we can assume without loss of generality that we study bifurcation of trivial solution $x = 0$ and so $F(\mu, 0) = 0$.

Consider the equation $\dot{\vec{x}} = \vec{F}(\mu, x)$ with $\vec{F}(\mu, 0) = 0$ (solution $\vec{x} = \vec{0}$). The value of the parameter $\mu = \mu_e$ is called bifurcation value if there exists a non-trivial solution in each nbd of $(\mu_e, \vec{0})$ in $R^m \times R^n$.

Example-18.2:

Consider the equation

$\dot{x} = 1 - 2(1 + \mu)x + x^2$ ……………………………….(18.3)

The equation

$$1 - 2(1 + \mu)x + x^2 = 0.$$

has the unique solution $x(\mu)$ if $-2(1 + \mu) + 2x \neq 0$ $i.e., if$ $x \neq 1 + \mu$. Equation (6.3) has the equilibrium solution

$x = 1 + \mu \pm \sqrt{2\mu + \mu^2}$ if $\mu \leq -2$ and $\mu > 0$.

Bifurcation can take place if

$$1 + \mu = 1 + \mu \pm \sqrt{2\mu + \mu^2}\ i.e., if\ \mu = 0 \text{ and } \mu = 2.$$

18.2: Normalisation:

Let us consider equations of the form

$\dot{\vec{x}} = A\vec{x} + \vec{f}(\vec{x})$ ……………………………….(18.4)

with A, a constant $n \times n$ matrix; $\vec{f}(\vec{x})$ can be expanded in the homogeneous vector polynomials which start with degree 2.

Let, $\vec{f}(\vec{x}) = \overrightarrow{f_2}(\vec{x}) + \overrightarrow{f_3}(\vec{x}) + \cdots$, the vector polynomial $\overrightarrow{f_m}(\vec{x}), m \geq 2$.

Consider the terms of the form $x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}$ where $m_1 + m_2 + \cdots + m_n = m$.

If λ is constant then

$$\overrightarrow{f_m}(\lambda\vec{x}) = \lambda^m \overrightarrow{f_m}(\vec{x}).$$

If, we are introduced in the behaviour of the solution in a nbd of critical points $\vec{x}=\vec{0}$, it is useful to introduce mean-identity transformation which simplified the vector function $\vec{f}(\vec{x})$. Even better, we find smooth transformation which turned equation (18.4) into linear equation. However linearization by transformation is most case is not possible.

Example-18.3:

Consider the equation

$\dot{x}=\lambda x+a_2x^2+a_3x^3+\cdots$ ....(18.5)

with $\lambda\neq 0, x\in R$.

We introduce the mean-identity transformation in the form of a series

$x=y+\alpha_2y^2+\alpha_3y^3+$ ....(18.6)

where we try to determine the coefficient $\alpha_2,\alpha_3,\ldots$ such that the equation for y is Linear. If we can determine this coefficients, the transformation (18.6) represents a formal expansion w.r.to y. This may be convergent for $y=0$. Substituting (18.6) into (18.5) we get

$$\dot{y}(1+\alpha_2y+3\alpha_3y^2+\cdots)=\lambda y+(\lambda\alpha_2+a_2)y^2+(\lambda\alpha_3+2a_2\alpha_2+a_3)y^3+\cdots$$

$$\text{or, }\dot{y}=\lambda y+(a_2-\lambda\alpha_2)y^2+(a_3+2\lambda\alpha_2^2-2\lambda\alpha_3)y^3+\cdots$$

For linearization we must have,

$$\alpha_2=\frac{a_2}{\lambda},\alpha_3=\frac{a_2^2}{\lambda^2}+\frac{a_3}{2\lambda}.$$

By this choice of α_2 *and* α_3, equation (18.5) is normalised to degree 3.

In the dimension of the equation is higher than one, the theory becomes more complicate. Equations (6.4) in the form

$$\dot{\vec{x}}=A\vec{x}+\vec{f_2}(\vec{x})+\vec{f_3}(\vec{x})+\cdots$$

will be transformed by

$\vec{x}=\vec{y}+\vec{h}(\vec{y})$ ....(18.7)

with $\vec{h}(\vec{y})$ consisting of a, probably infinite sum of homogeneous vector polynomials $\overrightarrow{h_m}(\vec{y}), m\geq 2$. So we can write transformation of (18.7) as

$$\vec{x}=\vec{y}+\overrightarrow{h_2}(\vec{y})+\overrightarrow{h_3}(\vec{y})+\cdots$$

We would like to determine $\vec{h}(\vec{y})$ such that $\dot{\vec{x}}=A\vec{y}$. Substituting (18.7) into (18.4) we get,

$$\dot{\vec{x}} = \dot{\vec{y}} + \frac{\partial \vec{h}}{\partial \vec{y}}\dot{\vec{y}} = (I + \frac{\partial \vec{h}}{\partial \vec{y}})\dot{\vec{y}}$$

$$= A\left(\vec{y} + \vec{h}(\vec{y})\right) + \vec{f}\left(\vec{y} + \vec{h}(\vec{y})\right)$$

Inversion of $(I + \frac{\partial \vec{h}}{\partial \vec{y}})$ in a nbd of $\vec{y} = \vec{0}$ yields (gives)

$$\dot{\vec{y}} = \left[I + \frac{\partial \vec{h}}{\partial \vec{y}}\right]^{-1} \quad [A\vec{y} + A\vec{h}(\vec{y}) + \vec{f}\left(\vec{y} + \vec{h}(\vec{y})\right)]$$

We start with removing the quadratic terms in (6.8). Expansion of $\vec{h}$ and $\vec{f}$ yields the equation

$$\frac{\partial \overrightarrow{h_2}}{\partial \vec{y}} A\vec{y} - A\overrightarrow{h_2} = \vec{f_2}(\vec{y})$$

This is the homology equation for $\overrightarrow{h_2}$. Similarity, the terms of degree m vanish, we have the homology equation

$\frac{\partial \overrightarrow{h_m}}{\partial \vec{y}} A\vec{y} - A\overrightarrow{h_m} = \vec{g}_m(\vec{y}), m \geq 2.$ ……………………………(18.9)

For $m > 2$, the right hand side $\vec{g}_m(\vec{y})$ can be expanded in terms of the solutions of the homology equation to degree mf. In considering the solvability of the homology equation (6.9) we observe that the left hand side is linear in $\overrightarrow{h_m}$. The linear mapping is given by $\vec{L}_A(\vec{h}) = \frac{\partial \vec{h}}{\partial \vec{y}} A\vec{y} - A\overrightarrow{h_2}(\vec{y})$

Carries homogenous vector polynomials even in vector polynomials of the same degree. If the set of polynomials$\vec{L}_A$ does not contain zero, $\vec{L}_A$ is invertible and equation (18.9) can be solved. For simplicity, we assume that all eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix A is in diagonal form. Written out in components $\vec{h}_m = (\vec{h}_{m_1}, \vec{h}_{m_2}, \dots, \vec{h}_{m_n})$, we have from (18.9)

$\sum_{j=1}^{n} \frac{\partial \vec{h}_{m_i}}{\partial \vec{y}_j} \lambda_j y_j - \lambda_i h_{m_i}(\vec{y}) = \vec{g}_{m_i}(\vec{y}), (i = 1,2, \dots, n; m \geq 2).$ ……….(18.10)

The term in h_{m_i} are all of the form

$$a y_1^{m_1} y_2^{m_2} \dots y_n^{m_n} = a y^{(m)}$$

where $m = m_1, m_2, \dots, m_n$, a constant. The eigen value of A are e_i and those of $\vec{L}_A$ are $y^{(m)} e_i$ with eigen values $\sum_{j=1}^{n} m_j\, y_j - \lambda_i (i = 1,2, \dots, n)$. If an eigen values $\vec{L}_A$ is zero, we call thisis resonance. If there is no resonance, equation (18.10) can be solved and the non-linear terms in (18.8) can be removed.

We summarise the above results in the following theorem-

Theorem-18.1: Poincare′ Theorem:

If the matrix A are non-resonant, the equation

$$\dot{\vec{x}} = A\vec{x} + \overrightarrow{f_2}(\vec{x}) + \overrightarrow{f_3}(\vec{x}) + \cdots$$

can be transformed into the linear equation $\dot{\vec{y}} = A\vec{y}$ by the transformation

$$\vec{x} = \vec{y} + \overrightarrow{h_2}(\vec{y}) + \overrightarrow{h_3}(\vec{y}) + \cdots$$

18.3: Centre Manifolds:

Consider the equation

$\dot{\vec{x}} = A\vec{x} + \vec{f}(\vec{x})$ ………………………….(18.11)

where A is constant $n \times n$ matrix and has eigen values with non-vanishing real part. The point $\vec{x} = \vec{0}$ to the corresponding manifolds of the equation $\dot{\vec{y}} = A\vec{y}$. If, one linearization we find eigen values zero or purely imaginary, bifurcation may arise. To study these phenomenon, we state the following theorem

Theorem-18.2:

Consider the equation

$\dot{\vec{x}} = A\vec{x} + \vec{f}(\vec{x})\ with\ \vec{x} \in R^n$ and A is constant $n \times n$ matrix; $\vec{x} = \vec{0}$ is belated critical point. The vector function $\vec{f}(\vec{x})\ is\ c^k, k \geq 2$, if in a nbd of $\vec{x} = \vec{0}$ and $\lim_{\|\vec{x} \to 0\|} \frac{\|\vec{f}(\vec{x})\|}{\|\vec{x}\|} = 0.$ The stable and unstable manifolds of equation $\dot{\vec{y}} = A\vec{y}$ are E_s and E_k, the space of eigen vectors corresponding with eigen values with zero real part is E_c. There exists e^k stable and unstable invariant manifolds E_s and E_k in $\vec{x} = \vec{0}$. There exists a c^{k-1} invariant manifolds W_c, called centre manifold which is tangent to E_c in $\vec{x} = \vec{0}$; if $k = \infty$, then W_c is in general c^m with $m \leq \infty$.

18.4 Bifurcations in One-Dimensional Systems:

The dynamics of a vector field on the real line of a system is very restricted as we have seen in the preceding chapters. However, the dynamics in one-dimensional systems depending upon parameters is interesting and have wide applications in science and engineering. Consider a one-dimensional continuous system

$$\dot{x}(t) = f(x, \mu); \quad x, \mu \in \mathbb{R} \tag{18.12}$$

depending on the parameter μ, where $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function of x and μ. The equilibrium points of (6.1) are the solutions of the equation

$$f(x, \mu) = 0.1 \tag{18.13}$$

The Eq. (18.13) clearly indicates that all the equilibrium points of the system (18.12) depend on the parameter μ, and they may change their stabilities as μ varies. Thus, bifurcations of a one-dimensional system are associated with the stabilities of its equilibri um points. Such bifurcations are known as local bifurcations as they occur in the neighbourhood of the equilibrium points. Such types of bifurcations are occurred in the population growth model, outbreak insect population model, chemical kinetics model, bulking of a beam, etc. In the following subsections, three important bifurcations, namely the saddle-node, pitchfork, and transcritical bifurcations are discussed in depth for one-dimensional systems.

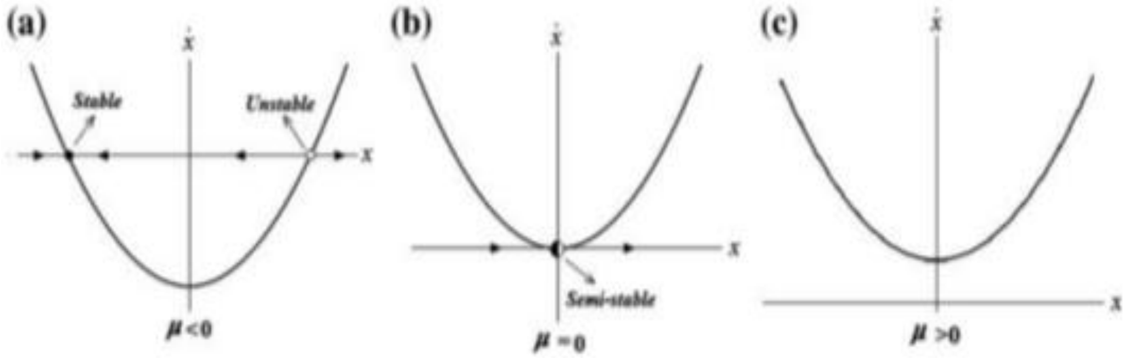
18.4.1 Saddle-Node Bifurcation:

Consider the one-dimensional system

$$\dot{x}(t) = f(x, \mu) = \mu + x^2; \quad x \in \mathbb{R} \tag{18.14}$$

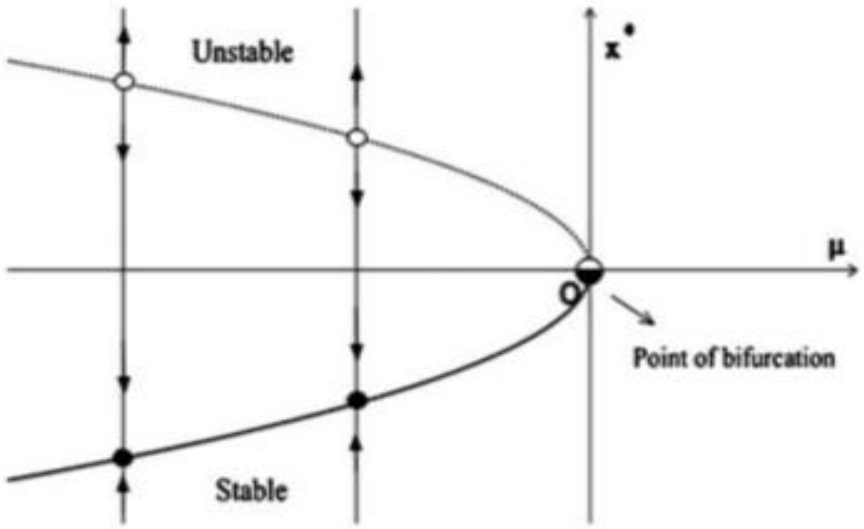
with μ as the parameter. Equilibri um points of (6.3) are obtained as

$$f(x, \mu) = 0 \Rightarrow \mu + x^2 = 0 \Rightarrow x^2 = -\mu \tag{18.15}$$



Depending upon the sign of the parameter μ, we have three possibilities. When $\mu < 0$, the system has two fixed points, $x_{1,2}^* = \pm\sqrt{-\mu}$. They merge at $x^* = 0$ when $\mu = 0$ and disappear when $\mu > 0$. We shall now analyze the system's behavior under flow consideration in the real line. The system $\dot{x} = f(x, \mu)$ represents a vector field $f(x, \mu)$ on the real line and gives the velocity vector x at each position x of the flow. As we discussed earlier, arrows point to the right direction if $x > 0$ and to the left if $x < 0$. So, the flow is to the right direction when $x > 0$ and to the left when $\dot{x} < 0$. At the points where $\dot{x} = 0$, there are no flows and such point s are called fixedpoints or equilibrium points of the system (18.14). The graph of the vector filed $f(x, \mu)$ in the $x - \dot{x}$ plane represents a parabola, as shown in Fig.

When $\mu < 0$, there are two fixed points of the system and are shown in Fig. 6.la. According to the flow imagination, the figure indicates that the fixed point at $x = \sqrt{-\mu}$ is unstable, whereas the fixed point at $x = -\sqrt{-\mu}$ is stable. From the figure, we also see that when μ approaches to zero from blow, the parabola moves up and the two fixed points move toward each other and they merge at $x = 0$ when $\mu = 0$. There are no fixed points of the system for $\mu > 0$, as shown in Fig. 6.1c. This is a very simple system but its dynamics is highly interesting. The bifurcation in the dynamics occurred at $\mu = 0$, since the vector fields for $\mu < 0$ and $\mu > 0$ are qualitatively different. The diagram of the parameter μ versus the fixed point x^* is known as the bifurcation diagram of the system and the point $\mu = 0$ is called the bifurcation point or the tuming point of the trajectory of the system. The bifurcation diagram is shown in Fig.



This is an example of a saddle-node bifurcation even though the system is one-dimensional. Actually, it is a subcritical saddle-node bifurcation, since the fixed points exist for values of the parameter below the bifurcation point $\mu = 0$. Consider another simple one-dimensional system

$$\dot{x} = \mu - x^2; \quad x, \mu \in \mathbb{R} \tag{18.16}$$

with parameter μ. This system can be obtained from (18.14) under the transformation $(x, \mu) \mapsto (-x, -\mu)$. So, the qualitative behaviour of the system (18.16) is just as the opposite of (18.14). Hence the system (18.16) has two equilibrium points $x_{1,2}^* = \pm\sqrt{\mu}$ for $\mu > 0$, they merge at $x^* = 0$ when $\mu = 0$ and disappear for $\mu < 0$. Thus, the qualitative behaviour of (18.16) is changing as μ passes through the origin. Hence $\mu = 0$ is the bifurcation point of the system (18.16). This is an example of a supercritical saddle-node bifurcation, since the equilibrium points exist for values of μ above the bifurcation point$\mu = 0$. The name 'saddle-node bifurcation' is not properly given because the actual bifurcation that occurred in this one-dimensional system is inconsistent with the name "saddle-node." The name is coined in comparison to the bifurcation pattem in two-dimensional systems in which a saddle and a node coincide and then disappear as the parameter exceeds the critical value. The saddle-node bifurcation in a one-dimensional system is connected with appearance and disappearance (vice versa) of the fixed points of the system as the parameter exceeds the critical value.

18.4.2 Pitchfork Bifurcation:

We now discuss pitchfork bifurcation in a cne-dimensional system which appears when the system has symunetry between left and right directions. In such a system, the fixed points tend to appear and disappear in symmetrical pair. For example, consider the one-dimensional system

$$\dot{x}(t) = f(x,\mu) = \mu x - x^3; \quad x,\mu \in \mathbb{R} \tag{18.17}$$

Replacing x by $-x$ in (18.17), we get

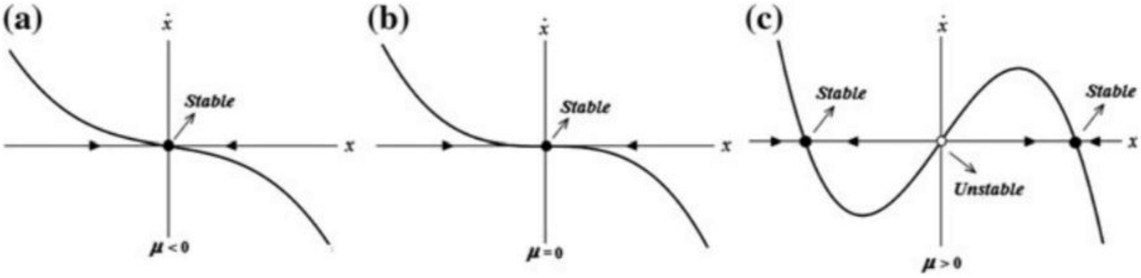
$$\begin{aligned} -\dot{x} &= -\mu x + x^3 = -(\mu x - x^3) \\ \Rightarrow \dot{x} &= \mu x - x^3 \end{aligned}$$

Thus the system is invariant under the transformation $x \mapsto -x$. The equilibrium points of the system are obtained as

$$f(x,\mu) = 0 \Rightarrow \mu x - x^3 = 0 \Rightarrow x = 0, \pm\sqrt{\mu}.$$

For $f(x,\mu) = \mu x - x^3$,

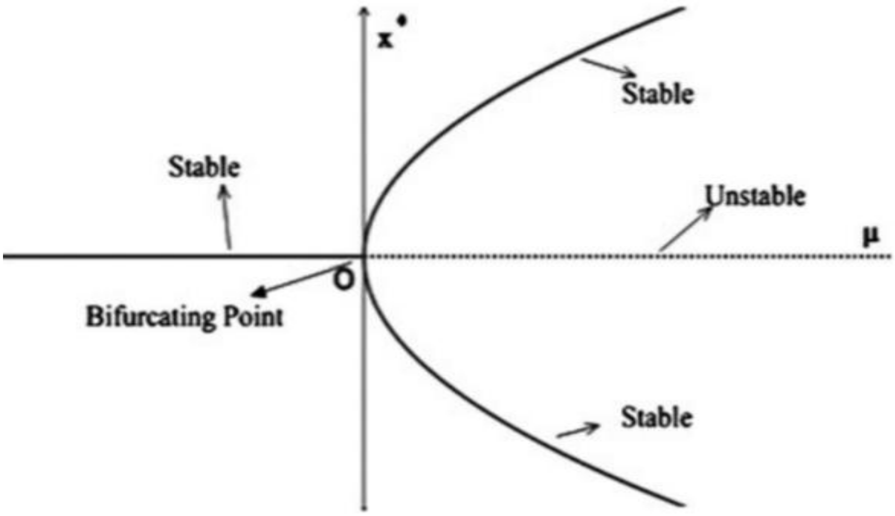
$$\frac{\partial f}{\partial x}(x,\mu) = \mu - 3x^2, \frac{\partial f}{\partial x}(0,\mu) = \mu, \frac{\partial f}{\partial x}(\pm\sqrt{\mu},\mu) = -2\mu.$$



When $\mu = 0$, the system has only one equilibrium point $x^* = 0$ and it is a equilibrium point in nature, since $(0,0) = 0$. For $\mu > 0$, three equilibrium points occur at $x^* = 0, \pm\sqrt{\mu}$, in which the equilibrium point origin $(x^* = 0)$ is a source (unstable) and the other two equilibrium points are sink (stable). For $\mu < 0$, the system has only one stable equilibrium point at the origin. The phase diagram in the $x - \dot{x}$ plane is depicted in Fig.

From the diagram we see that when μ increases from negative to zero, the equilibrium point origin is still stable but much more weakly, because of its nonhyperbolic nature. When $\mu > 0$, the origin becomes unstableequilibrium point and two new stable equilibrium points appear on either side of the origin located at $x = -\sqrt{\mu}$ and $x = \sqrt{\mu}$. The bifurcation diagram of the system is shown in Fig. 6.4. From the pitchfork-shape bifurcation diagram, the name 'pitchfork' becomes clear. But it is basically a pitchfork trifurcation of the system. The bifurcation for this vector field is called a supercritical pitchfork bifurcation, in which a stable equilibrium bi furcates into two stable equilibria. Transforming (x, μ) into $(-x, -\mu)$, we can directly obtain another pitchfork bifurcation, the subcritical pitchfork bifurcation, described by the system

$$\dot{x}(t) = \mu x + x^3. \qquad (18.18)$$



This system has three equilibrium points $x^* = 0, \pm\sqrt{-\mu}$ for $\mu < 0$, in which the equilibrium point $x^* = 0$ is stable and the other two are unstable. For $\mu > 0$, it has only one equilibrium point $x^* = 0$, which is unstable.

18.4.3 Transcritical Bifurcation:

There are many parameter-dependent physical systems for which an equilibrium point must exist for all values of a parameter of the system and can never disappear. But it may change its stability character as the parameter varies. The transcritical bifurcation is one such type of bifurcation in which the stability characters of the fixed points are changed for varying values of the parameters. Consider the one-dimensional system

$$x = f(x, \mu) = \mu x - x^2; \quad x \in \mathbb{R} \qquad (18.19)$$

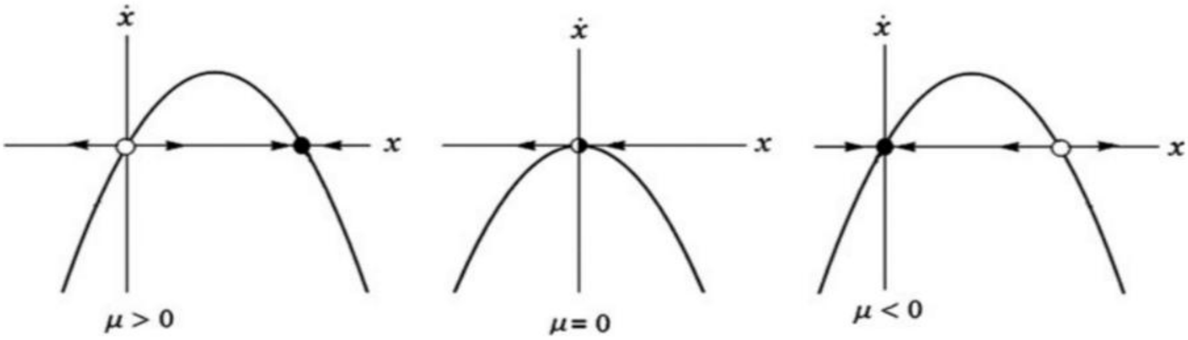
with $\mu \in \mathbb{R}$ as the parameter. The equilibrium points of this system are obtained as

$$f(x, \mu) = 0 \Rightarrow \mu x - x^2 = 0 \Rightarrow x = 0, \mu.$$

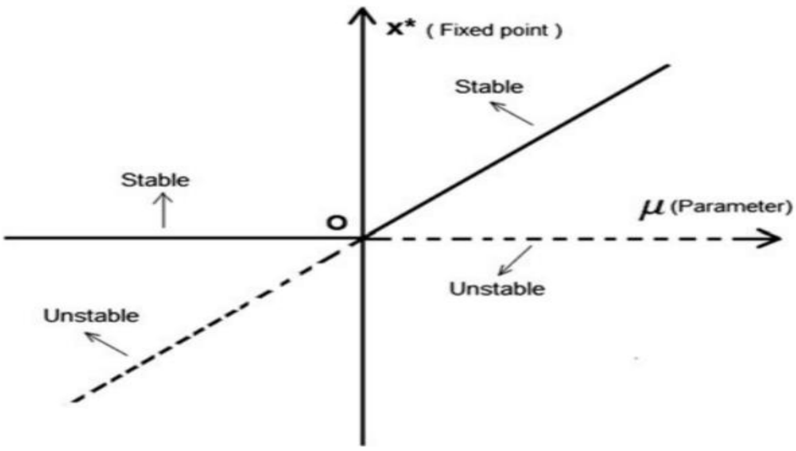
Thus the system has two equilibrium points $x^* = 0, \mu$. We calculate

$$\frac{\partial f}{\partial x}(x, \mu) = \mu - 2x, s0, \frac{\partial f}{\partial x}(0, \mu) = \mu, \frac{\partial f}{\partial x}(\mu, \mu) = -\mu.$$

This shows that for $\mu = 0$ the system has only one equilibrium point at $x^* = 0$, which is nonhyperbolicequilibrium points. For $\mu \neq 0$, it has two distinct equilibrium points $x^* = 0, \mu$, in which the equilibrium point origin is a source (unstable) for $\mu > 0$ and it is a sink (stable) for $\mu < 0$. The other equilibrium point $x^* = \mu$ is unstable if $\mu < 0$ and stable for $\mu > 0$. The phase diagrams for the above three cases are shown in Fig.



This type of bifurcation is known as transcritical bifurcation. In this bifurcation, an exchange of stabilities has taken place between the two fixed points of the system. The bifurcation diagram is presented in Fig.



18.5: Bifurcation of Equilibrium Solutions Hopf Bifurcation:

Consider the equation

$\dot{\vec{x}} = A(\mu)\vec{x} + \vec{f}(\mu, \vec{x})$ …………………(18.20)

$with\ \vec{x} \in R^n$, μ-being a parameter in R. We suspend this n-dimensional system in a (n+1)-dimensional system by adding μ as new variable:

$$\dot{\vec{x}} = A(\mu)\vec{x} + \vec{f}(\mu, \vec{x})$$

$$\dot{\mu} = 0. \qquad \text{………….…………..(18.21)}$$

Suppose, now that $\frac{\partial \vec{f}}{\partial \vec{x}} \rightarrow \vec{0}, as\ \|\vec{x}\| \rightarrow 0$ and consider the possibility of bifurcation of the solution $\vec{x} = \vec{0}$. If the matrix A has p-eigen values with zero real parts for a certain value of μ, equation (18.21) has (p+1)-dimensional centre manifold W_c. We study W_c, we simplify the equation by normalisation.

Example-6.4:

Consider the system

$$\dot{x} = \mu x - x^3 + xy$$

$$\dot{y} = -y + y^2 - x^2 \qquad \text{………………..(18.22)}$$

We shall consider the bifurcation in a nbd of (0, 0) for small values of $|\mu|$. We write the system (18.22) as

$$\dot{x} = \mu x - x^3 + xy$$

$$\dot{y} = -y + y^2 - x^2$$

$$\dot{\mu} = 0. \qquad \text{…………..…………..(18.23)}$$

The system, linearized in nbd of (0,0,0) has the eigen values $(\mu, -1, 0)$. So according to theorem (18.2), there exists a two dimensional centre manifold $y = h(x, \mu)$.

Differentiating and by using the system (18.23) we find

$$\frac{\partial h}{\partial x}(\mu x - x^3 + xh) = -h + h^2 - x^2$$

$$\dot{\mu} = 0$$

Also a Taylor expansion for h w.r.to $x \; and \; \mu$ gives

$$h(x, \mu) = -x^2 + \cdots$$

In the centre manifold, the flow is determined by the equation

$$\dot{u} = \mu u - 2u^3 + \cdots$$

$$\dot{\mu} = 0.$$

<u>The Saddle-Node Bifurcation:</u>

In the centre manifold, the flow is described by $\dot{u} = \mu - u^2 \; and \; \dot{\mu} = 0$. If $\mu < 0$, there exists no equilibrium solution. At $\mu = 0$, two equilibrium solutions branch off, one of which is stable and the other unstable.

The Trans-Critical Bifurcation:

The flow is described by the equation $\dot{u} = \mu - u^2 \; and \; \dot{\mu} = 0$. Apart from (0, 0) there are always two equilibrium solutions with an change of stability when passing through $\mu = 0$.

The Pitch-Fonk Bifurcation:

The flow is described by the equation $\dot{u} = \mu x - u^3 \; and \; \dot{\mu} = 0$. If $\mu \leq 0$, there is one equilibrium solution which is stable. If $\mu > 0$, there are three equilibrium solutions of which $u = 0$ is unstable, the two solutions which have branched off at $\mu = 0$ are stable. This is called pitch-fonk bifurcation and it is supercritical.

On replacing $-u^3 \; by + u^3$, the figure is reflected w.r.to the u-axis in this case the bifurcation is subcritical.

Bifurcation of Periodic Solution(Hopf-Bifurcation):

Consider the system

$$\dot{x} = \mu x - \omega y + \cdots$$

$$\dot{y} = \omega x + \mu y + \cdots$$

where $\omega(\neq 0)$ is fixed. If $\mu = 0$, the eigen values of linear part are purely imaginary. Normalisation remove all quadratic and a number of cubic terms. To degree three, the normal form of above equation

$$\dot{u} = d\mu u - (\omega + c\mu)v + a(u^2 + v^2)u - b(u^2 + v^2)v + \cdots$$

$$\dot{u} = (\omega + c\mu)u + d\mu v + b(u^2 + v^2)u + a(u^2 + v^2)v$$

In polar co-ordinate the system is

$$\dot{r} = (d\mu + ar^2)r + \cdots$$

$\dot{\theta} = \omega + c\mu + br^2 + \cdots$ …………..(18.24)

At $\mu = 0$, we have a pitch-fonk bifurcation of amplitude(r) equation which corresponds with a Hopf-bifurcation for the full system. A periodic solution of (18.24) exists if $d \neq 0, a \neq 0$ with amplitude $= \left(-\frac{d\mu}{a}\right)^{\frac{1}{2}}$.

Exercises:

1 What do you mean by 'bifurcation' of a system?

2 Formulate one physical system in which bifurcation occurs for changing values of the parameter. Draw also bifurcation diagram.

3 Find the critical value of μ in which bifurcation occurs for the following systems:
(i)

$\dot{x} = \mu x + x^2$, (ii) $\dot{x} = 1 + \mu x + x^2$, (iii) $\dot{x} = \mu x + x^3$, (iv) $\dot{x} = \mu x - x^3$,
$\dot{x} = x^2 - \mu, \quad \mu \in \mathbb{R}$, (v) $\dot{x} = \mu x + x^2 - x^3 + x^4$, (vi) $\dot{x} = -\mu_1 x - \mu_2 x^2$.

4 Determine the bifurcation point (μ_0, x_0) for the system $\dot{x} = \mu x - cx^2, (c \neq 0)$.
Sketch the phase portraits for $\mu < \mu_0$ and $\mu > \mu_0$.

UNIT-19

Bifurcation of equilibrium solutions – the saddle node bifurcation, the pitch-fork bifurcation, Hopf-bifurcation.

19.1 Introduction:

The dynamics of a continuous system $\dot{x} = f(x,\mu)$ depends on the parameter$\mu \epsilon R$. It is often found that μ crosses a critical value, the properties of dynamical evolution, e.g., its stability, fixed points, periodicity etc. may change. Moreover, a completely new orbit may be created. Basically, a structurally unstable system is termed as bifurcation. The bifurcation diagram is very useful in understanding the dynamical behaviour of a system. Bifurcations associated with a single parameter are called codimension-1 bifurcations. On the other hand, bifurcations connected with two parameters are known as codiemension-2 bifurcations. These bifurcations give many interesting dynamics and have a wide range of applications in biological and physical sciences. Various bifurcations and their theories are the integral part of nonlinear systems. We discuss some important bifurcations in one- and two dimensional systems in the following sections.

19.2 Bifurcations in One-Dimensional Systems: A General Theory:

So far we have discussed bifurcations based on the flow of vector fields. We now derive a general mathematical theory for bifurcations in one-dimensional systems. Consider a general one-dimensional system

$$\dot{x}(t) = f(x,\mu); \quad x,\mu \in \mathbb{R} \tag{19.1}$$

where $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function. If μ_0 be the bifurcation point and x_0 be the corresponding equilibrium point of the system, then x_0 is nonhyperbolic if

$$\frac{\partial f}{\partial x}(x_0,\mu_0) = 0. \tag{19.2}$$

We first establish the condition for the saddle-node bifurcation.

19.2.1 Saddle-Node Bifurcation:

We assume that

$$\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0 \qquad (19.3)$$

Then by the implicit function theorem, there exists a unique smooth function $\mu = \mu(x)$ with $\mu(x_0) = \mu_0$, in the neighborhood of (x_0, μ_0) such that $f(x, \mu(x)) = 0$. Differentiating the equation $f(x, \mu(x)) = 0$ with respect to x, we have

$$0 = \frac{\mathrm{d}f}{\mathrm{d}x}(x, \mu(x)) = \frac{\partial f}{\partial x}(x, \mu(x)) + \frac{\partial f}{\partial \mu}(x, \mu(x))\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) \qquad (19.4)$$

Therefore, at (x_0, μ_0), we get

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0, \mu_0) + \frac{\partial f}{\partial \mu}(x_0, \mu_0) \quad \frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0) &= 0 \\ \Rightarrow \frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0) &= -\frac{\frac{\partial}{\partial x}(x_0,\mu_0)}{\frac{\partial f}{\partial \mu^2}(x_0,\mu_0)} \qquad (19.5) \\ \Rightarrow \frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0) &= 0 \end{aligned}$$

Again, differentiating the equation $f(x, \mu(x)) = 0$ with respect to x twice, we get

$$\begin{aligned} 0 = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x, \mu(x)) = &\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial x}(x, \mu(x)) + \frac{\partial f}{\partial \mu}(x, \mu(x))\frac{\mathrm{d}\mu}{\mathrm{d}x}(x)\right) \\ = &\ \frac{\partial^2 f}{\partial x^2}(x, \mu(x)) + 2\frac{\partial^2 f}{\partial x\,\partial \mu}(x, \mu(x))\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) + \frac{\partial^2 f}{\partial^2 \mu}(x, \mu(x))\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}(x)\right)^2 \\ &+ \frac{\partial f}{\partial \mu}(x, \mu(x))\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2}(x) \end{aligned}$$

(19.6)

Therefore, at (x_0, μ_0),

$$\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) + \frac{\partial f}{\partial \mu}(x_0, \mu_0)\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2}(x_0) = 0. \qquad (19.7)$$

Now, recall the saddle-node bifurcation diagram of the system. In this diagram, the unique curve $x = \mu^2$ of fixed points, passing through (0,0), lies entirely on only one side of the bifurcation point $\mu = 0$. This will be possible only if

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} \neq 0$$

in the neighborhood of $(x, \mu) = (0,0)$. Compared to this, we take

$$\frac{\mathrm{d}^2\mu}{\mathrm{dx}}(x_0) \neq 0.$$

Therefore from (19.7), we see that

$$\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0. \qquad (19.8)$$

We state the result in the following theorem.

Theorem 19.1 (Saddle-node bifurcation) Suppose the system $\dot{x}(t) = f(x, \mu), x, \mu \in \mathbb{R}$ has an equilibrium point $x = x_0$ at $\mu = \mu_0$ satisfying the conditions

$$f(x_0, \mu_0) = 0, \frac{\partial f}{\partial x}(x_0, \mu_0) = 0.$$

If

$$\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0, \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0,$$

then the system has a saddle-node bifurcation at (x_0, μ_0). Similarly, one can easily derive the conditions under which the system $\dot{x}(t) = f(x, \mu), x, \mu \in \mathbb{R}$ possess transcritical and pitchfork bifurcations. In this book, we only state the following theorem for these two bifurcations.

Theorem 19.2 (Transcritical and pitchfork bifurcations) Suppose the system $\dot{x}(t) = f(x, \mu), x, \mu \in \mathbb{R}$ has an equilibrium point $x = x_0$ at $\mu = \mu_0$ satisfying the conditions

$$f(x_0, \mu_0) = 0, \frac{\partial f}{\partial x}(x_0, \mu_0) = 0.$$

(i) If

$$\frac{\partial f}{\partial \mu}(x_0, \mu_0) = 0, \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0 \text{ and } \frac{\partial^2 f}{\partial x \, \partial \mu}(x_0, \mu_0) \neq 0$$

then the system has a transcritical bifurcation at (x_0, μ_0).

(ii) If

$$\frac{\partial f}{\partial \mu}(x_0, \mu_0) = 0, \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) = 0, \frac{\partial^2 f}{\partial x \, \partial \mu}(x_0, \mu_0) \neq 0 \text{ and } \frac{\partial^3 f}{\partial x^3}(x_0, \mu_0) \neq 0,$$

then the system has a pitchfork bifiurcation at (x_0, μ_0).
We now derive the nomal forms of these bifurcations in one-dimensional systems. By normal

form of a system we mean the most simplified mathematical form from which one can easily understand the type of bifurcations occurred in the system.

(a) **Normal form of saddle-node bifurcation:**

Suppose the system has an equilibrium point at $x = x_0$ for $\mu = \mu_0$ for which all the saddle-node bifurcation conditions are satisfied, that is,

$$f(x_0, \mu_0) = 0, \frac{\partial f}{\partial x}(x_0, \mu_0) = 0, \frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0, \tag{19.9}$$

Expanding $f(x, \mu)$ in a Taylor series in the neighborhood of (x_0, μ_0), we have

$$\begin{aligned} \dot{x} = &\ f(x, \mu) \\ = &\ f(x_0, \mu_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, \mu_0) + (\mu - \mu_0)\frac{\partial f}{\partial \mu}(x_0, \mu_0) + \frac{1}{2!}(x - x_0)^2 \frac{\partial f}{\partial x^2}(x_0, \mu_0) \\ &+ (x - x_0)(\mu - \mu_0)\frac{\partial^2 f}{\partial \pi\, \partial\, \partial_0}(x_0, \mu_0) + \frac{1}{2!}(\mu - \mu_0)^2 \frac{\partial^2 f}{\partial \mu^2}(x_0, \mu_0) + \cdots \\ = &\ (\mu - \mu_0)\frac{\partial f}{\partial \mu}(x_0, \mu_0) + \frac{1}{2!}(x - x_0)^2 \frac{\partial f}{\partial x^2}(x_0, \mu_0) + \cdots \\ = &\ \alpha(\mu - \mu_0) + \beta(x - x_0)^2 + \cdots \qquad (19.10) \end{aligned}$$

where $\alpha = \frac{f}{\partial j}(x_0, \mu_0)$ and $\beta = \frac{1\alpha_j}{2z\alpha^2}(x_0, \mu_0)$ are nonzero real. The Eq. (19.10) refers to as the normal form of the saddle-node bifurcation. This is a great advantage for determining the bifurcation which a system undergoes.

(b) **Normal form of transcritical bifurcation:**

Suppose that the system has an equilibrium point $x = x_0$ at $\mu = \mu_0$ for which the transcritical bifurcation conditions are satisfied as given in Theorem 19.2(i). Using the Taylor series expansion of $f(x, \mu)$ in the neighborhood of (x_0, μ_0), we have

$$\begin{aligned} \dot{x} = &\ f(x, \mu) \\ = &\ f(x_0, \mu_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, \mu_0) + (\mu - \mu_0)\frac{\partial f}{\partial \mu}(x_0, \mu_0) + \frac{1}{2!}(x - x_0)^2 \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \\ &+ (x - x_0)(\mu - \mu_0)\frac{\partial^2 f}{\partial x\, \partial \mu}(x_0, \mu_0) + \frac{1}{2!}(\mu - \mu_0)^2 \frac{\partial^2 f}{\partial \mu^2}(x_0, \mu_0) + \cdots \\ = &\ (x - x_0)(\mu - \mu_0)\frac{\partial^2 f}{\partial x\, \partial \mu}(x_0, \mu_0) + \frac{1}{2!}(x - x_0)^2 \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) + \cdots \\ = &\ \alpha(x - x_0)(\mu - \mu_0) + \beta(x - x_0)^2 + \cdots \qquad (19.11) \end{aligned}$$

where $\alpha = \frac{\partial f}{\alpha q_j}(x_0, \mu_0)$ and $\beta = \frac{1/f}{2i}(x_0, \mu_0)$ are nonzero real. The Eq. (19.11) refers to the normal form of the transcritical bifurcation.

(c) Normal form of pitchfork bifurcation:

Suppose that the system has an equilibrium point $x = x_0$ at $\mu = \mu_0$, satisfying all the pitchfork bifurcation conditions given in Theorem 19.2(ii). We now expand the function $f(x,\mu)$ in the neighborhood of (x_0,μ_0) in Taylor series expansion as presented below.

$$
\begin{aligned}
\dot{x} = \;& f(x,\mu) \\
= \;& f(x_0,\mu_0) + (x-x_0)\frac{\partial f}{\partial x}(x_0,\mu_0) + (\mu-\mu_0)\frac{\partial f}{\partial \mu}(x_0,\mu_0) + \frac{1}{2}(x-x_0)^2\frac{\partial^2 f}{\partial x^2}(x_0,\mu_0) \\
& +(x-x_0)(\mu-\mu_0)\frac{\partial^2 f}{\partial x\,\partial \mu}(x_0,\mu_0) + \frac{1}{2}(\mu-\mu_0)^2\frac{\partial^2 f}{\partial \mu^2}(x_0,\mu_0) \\
& +\frac{1}{6}(x-x_0)^3\frac{\partial^3 f}{\partial x^3}(x_0,\mu_0) + \frac{1}{2}(x-x_0)^2(\mu-\mu_0)\frac{\partial^3 f}{\partial x^2\,\partial \mu}(x_0,\mu_0) \\
& +\frac{1}{2}(x-x_0)(\mu-\mu_0)^2\frac{\partial^3 f}{\partial x\,\partial \mu^2}(x_0,\mu_0) + \frac{1}{6}(\mu-\mu_0)^2\frac{\partial^2 f}{\partial \mu^2}(x_0,\mu_0) + \cdots \\
& = (x-x_0)(\mu-\mu_0)\frac{\partial^2 f}{\partial x\,\partial \mu}(x_0,\mu_0) + \frac{1}{6}(x-x_0)^3\frac{\partial^3 f}{\partial x^3}(x_0,\mu_0) + \cdots \\
& = \alpha(x-x_0)(\mu-\mu_0) + \beta(x-x_0)^3 + \cdots \qquad (19.12)
\end{aligned}
$$

of the pitchfork bifurcation.

Example 19. 1 Show that the system

$$\dot{x} = x(1-x^2) - a(1-e^{-bx})$$

undergoes a transcritical bifurcation at $x = 0$ when the parameters a and b satisfy a certain relation to be determined.

Solution; Clearly, $x = 0$ is an equilibrium point of the given system for all values of the parameters a and b. This indicates that the system will exhibit a transcritital bifurcation. For small x, the expansion of e^{-bx} gives

$$e^{-bx} = 1 - bx + \frac{b^2x^2}{2!} - O(x^3).$$

Therefore,

$$
\begin{aligned}
\dot{x} & = x(1-x^2) - a(1-e^{-bx}) \\
& = x(1-x^2) - a\left(bx - \frac{b^2x^2}{2!} + O(x^3)\right) \\
& = (1-ab)x + \frac{1}{2}ab^2x^2. \quad \text{[Neglecting cube andhigher powers of } x\text{]}
\end{aligned}
$$

For transcritical bifurcation at $x = 0, 1 - ab = 0$, that is, $ab = 1$. Hence the system undergoes a transcritical bifurcation at $x = 0$ when $ab = 1$.

Example 19.2: Describe the bifurcation of the system $\dot{x} = x^3 - 5x^2 - (\mu - 8)x + \mu - 4$.

Solution Let $f(x,\mu) = x^3 - 5x^2 - (\mu - 8)x + \mu - 4$. The equilibrium points are given by

$$x^3 - 5x^2 - (\mu - 8)x + \mu - 4 = 0$$
$$\Rightarrow (x-1)(x^2 - 4x - \mu + 4) = 0$$

Clearly, $x = 1$ is a fixed point of the system for all values of μ. The other two fixed points are $x_{\pm} = 2 \pm \sqrt{\mu}$, which are real and distinct for $\mu > 0$. They coincide with the fixed point $x = 2$ for $\mu = 0$ and vanish when $\mu < 0$. Therefore, the system has a saddle-node bifurcation at $x = 2$ with $\mu = 0$ as the bifurcation point. We can also verify this using Theorem 19.1. Take $x_0 = 2$ and $\mu_0 = 0$. Now, calculate

$$\frac{\partial f}{\partial x}(x,\mu) = 3x^2 - 10x + 8 - \mu, \frac{\partial f}{\partial \mu}(x,\mu) = -x + 1, \frac{\partial^2 f}{\partial x^2}(x,\mu) = 6x - 10.$$

We see that

$$f(x_0,\mu_0) = 0, \frac{\partial f}{\partial x}(x_0,\mu_0) = 0, \frac{\partial f}{\partial \mu}(x_0,\mu_0) = -1 \neq 0, \frac{\partial^2 f}{\partial x^2}(x_0,\mu_0) = 2 \neq 0.$$

Therefore, by Theorem 19.1 the system has a saddle-node bifurcation at (x_0,μ_0), where $x_0 = 2$ and $\mu_0 = 0$.

Example 19.3: Consider the system $\dot{x} = x^3 - \mu; x,\mu \in \mathbb{R}$. Does the system has any bifurcation in neighborhood of its fixed points? Justify.
Solution Here $f(x,\mu) = x^3 - \mu$. The fixed points are given by

$$f(x,\mu) = 0 \Rightarrow x^3 - \mu = 0 \Rightarrow x = \mu^{1/3}.$$

Calculate

$$\frac{\partial f}{\partial x}(x,\mu) = 3x^2, \text{ so } \frac{\partial f}{\partial x}\left(\mu^{1/3},\mu\right) = 3\mu^{2/3}.$$

The qualitative behavior of the system does not change with the variation of the parameter μ. So, bifurcation does not occur in the neighborhood of its fixed points.

19.3 Imperfect Bifurcation:

Consider the system represented by the Eq. (18.3). Suppose this system exhibits a saddle-node bifurcation at the point $(x,\mu) = (0,0)$. If we add a quantity $\varepsilon \in \mathbb{R}$ in this equation and then apply Theorem 19.1, we see that the system also has a saddle-node bifurcation at $(x,\mu) = (0,-\varepsilon)$. Thus an addition of the term ε in (18.3) does not change its bifurcation character. In similar way, addition of the term εx in the Eq. (18.3) will not produce any new bifurcation pattern, provided that the parameter $\mu \neq 0$. This bifurcation is structurally stable. The other two bifurcations, mentioned earlier, are not structurally stable. They can alter under arbitrarily small perturbations and produce new bifurcations. These bi furcations are called imperfect bifurcations and the parameter (perturbation quantity) is known as the imperfection parameter. For example, consider the system

$$\dot{x}(t) = \varepsilon + \mu x - x^2 \qquad (19.13)$$

where $\varepsilon, \mu \in \mathbb{R}$ are parameters. If $\varepsilon = 0$, it reduces to the system (6.8) and so, it has a transcritical bifurcation. We shall now analyze the system for $\varepsilon \neq 0$. The equilibrium points of (19.13) are the solutions of the equation

$$x = 0 \Rightarrow s + \mu x - x^2 = 0$$
$$\Rightarrow x = \frac{\mu \pm \sqrt{\mu^2 + 4\varepsilon}}{2}.$$

If $g < 0$, then (19.13) has two distinct equilibrium points

$$x_{\pm}^{*} = \frac{\mu \pm \sqrt{\mu^2 + 4\varepsilon}}{2}$$

when the parameter μ lies in $(-\infty, -2\sqrt{-8}) \cup (2\sqrt{-8}, \infty)$ in which x_{+}^{*} is stable and x_{-}^{*} is unstable. These two equilibrium points merge at $x^* = \mu/2$ when $\mu = \pm 2\sqrt{-\varepsilon}$ and disappear when μ lies in $(-2\sqrt{-\varepsilon}, 2\sqrt{-\varepsilon})$. Thus, for $\varepsilon < 0$, the transcritical bifurcation for $\varepsilon = 0$ perturbs into two saddle-node bifurcations at $(-\sqrt{-g}, -2\sqrt{-8})$ and $(\sqrt{-8}, 2\sqrt{-8})$ with bifurcation points $\mu = -2\sqrt{-8}$ and $\mu = 2\sqrt{-8}$, respectively.

Again, if $\varepsilon > 0$, then $(\mu^2 + 4\varepsilon) > 0$ for all μ. Therefore, in this case, the system has two distinct (nonintersecting) solution curves, one is stable and the other is unstable, and so no bifurcations will appear as μ varies. In conclusion, the addition of small quantity in a system will change the bifurcation character when the bifurcation pattem is not structurally stable.

19.4 Bifurcations in Two-Dimensional Systems:

Dynamics of two-dimension systems are vast and their qualitative behaviors are determined by the nature of equilibrium points, periodic orbits, limit cycles, etc. The parameters and their critical values for bifurcations are highly associated with system's evolution and have physical significances. The critical parameter value is a deciding factor for a system undergoing bifurcation solutions. We shall now formulate a simple problem where the critical value for qualitative change in the system can be obtained very easily. Consider a circular tube suspended by a string attached to its highest point and carrying a heavy mass m, which is rotating with an angular velocity ω about the vertical axis.

The angular motion of the mass m is determined by the following equation without taking into account the damping force,

$$\begin{aligned} ma\tilde{\theta} &= ma\omega^2 \sin\theta\cos\theta - mg\sin\theta \\ \text{or,} \quad \tilde{\theta} &= \left(\omega^2\cos\theta - \frac{g}{a}\right)\sin\theta, \end{aligned}$$

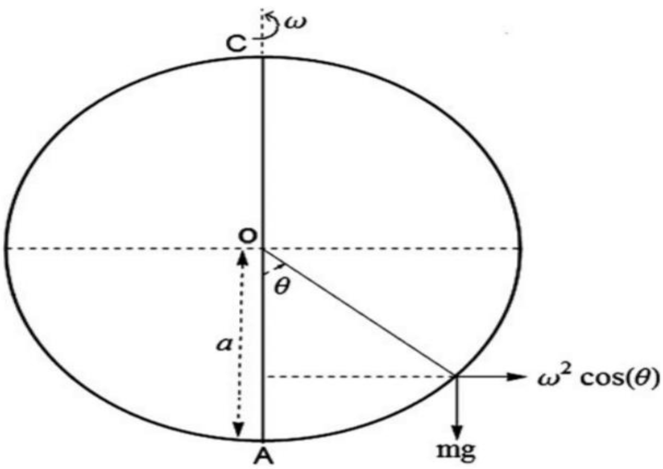
where a is the radius of the circular tube and g is the acceleration due to gravity and $\mu = \omega^2 a/g$. The right-hand side of the above equation may be denoted by

$$f(\theta,\mu) = \frac{g}{a}(\mu\cos\theta - 1)\sin\theta.$$

The equilibrium positions are given by $f(\theta,\mu) = 0$. Thus, there exist two positions of equilibrium and are given by$\sin\theta = 0 \Rightarrow \theta = 0, \pi, -\pi$ according to the problem and $\cos\theta = \frac{1}{\mu} = \frac{8}{\omega\omega^2}$.

If $\omega^2 < g/a$, that is, if $\mu < 1$, then $\cos\theta > 1$ and so $\theta = 0$ is the only position of equilibrium of the system, and it is stable for small θ,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{a}(1-\mu)\theta$$



Any small displacement, say $\theta = \theta_0$ with $\dot{\theta} = 0$ will result in small oscillations about the lowest point $\theta = 0$.

As ω increases beyond the critical value $\omega_\sigma > \sqrt{\frac{c}{a}}$, the equilibrium at $\theta = 0$ loses its stability and a new position of equilibrium $\theta = \cos^{-1}(g/a\omega^2)$ is created. This is a position of stable equilibrium. Thus, we see that a bifurcation occurs when the angular velocity ω crosses the critical value $\omega_{cr} = \sqrt{\frac{1}{a}}$, that is, $\mu = 1$.

This simple example illustrates how bifurcation occurs and how the behavior of the system alters before and after the bifurcating point. In the following subsections, we shall discuss few common bifurcations that frequently occur in two-dimensional systems, viz., (i) saddle-node bifurcation, (ii) transcritical bifurcation, (iii) pitchfork bifurcation, (iv) Hopf bifurcation, and (v) homoclinic and heteroclinic bifurcations.

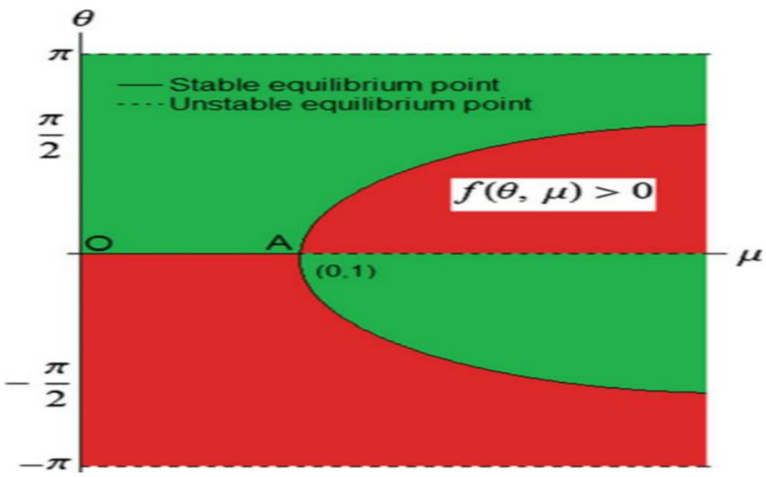
19.4.1 Saddle-Node Bifurcation:

Consider a parameter-dependent two-dimensional system

$$\dot{x} = \mu - x^2, \dot{y} = -y; \mu \in \mathbb{R} \qquad (19.14)$$

The fixed points of the system are the solutions of the equations

$$\dot{x} = 0, \dot{y} = 0$$



which yield

$$\mu - x^2 = 0, y = 0.$$

For $\mu > 0$, the Eq(19.14) hes two distinct fixed poimts at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$. These two fixed points merge at the origin $(0,0)$ when $\mu = 0$ and they varish when $\mu < 0$. This is a sarre feature as we heve seen in cine-dinensional sadik-node bifuxation. We shall now deternine the stabilitis of the fixed points. This need to evaluate the Jacohian matrix of the system for kical stakility behavior and is given by

$$J(x, y) = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

We first consider the case $\mu > 0$. Hex the sysken has two fixed points $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$. The Jacobian

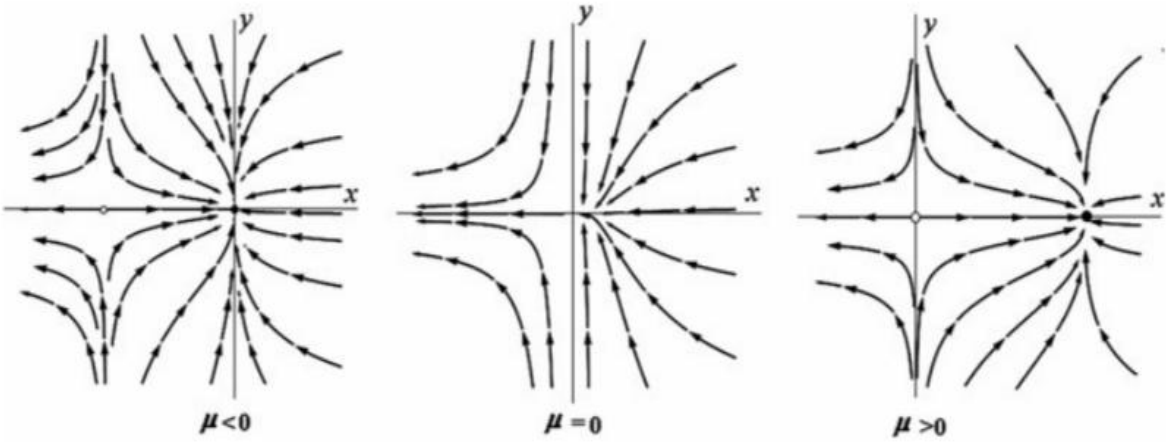
$$f(\sqrt{\mu}, 0) = \begin{pmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix}$$

at the fixed point $(\sqrt{\mu}, 0)$ has two eigenvahes $(-2\sqrt{\mu})$ and (-1), which are xal and negative. Hence the fixed point $(\sqrt{\mu}, 0)$ is a stable node. Sirnilarly, calculating the eigenvalues of $J(-\sqrt{\mu}, 0)$ we can show that the fixed pcimt $(-\sqrt{\mu}, 0)$ is saddle. Consider the second case, $\mu = 0$. In thas case, the syittem has a single fixed peint (0,0). The Jacoltian matrix at (0,0) is

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

with eigenvalues $0, (-1)$. This indicats that the fixed point (0,0) is semi-stable. For $\mu < 0$, the system has no fixed pcimts.

Thus we see that the system (19.14) has two fixed points, one is a stable node and the other is a saddle point, when $\mu > 0$. As μ decreasesi, the saddle and the stable node aprroach each other. They collide at $\mu = 0$ and disappear when $\mu < 0$. The phase portaits are shown for diffexent values of the parameter in Fig.



From the phase diagram we see that when the parameter is positive, no matter how small, all trajectories in the region $\{(x, y): x > -\sqrt{\mu'}\}$ reach steadily at the stable node origin of the system. As soon as μ crosses the origin, an exchenge of stability lakes place and this clearly indicates in the phase portrait of the system. When μ is negative, atl trajectaries eventuatly excape to infinity. This type of hifuatation is known as saddle-node hifurcation. The mame "saddle-node" is because its hasic mechanism is the cotlision of two fixed points, viz, a saddle and a node of the system. Here $\mu = 0$ is the bifuzation pcint. The hifurcation diagram is same as that for the onedimensional system.

19.4.2 Transcritical Bifurcation:

Consider a two-dimensional parametric system expressed by

$\dot{x} = \mu x - x^2, \dot{y} = -y;$ (19.15) with parimeler$\mu \in$ R. This system has always two distinct fixed points (0,0) and $(\mu, 0)$ for $\mu \neq 0$. For $\mu = 0$, these two fioed poimts merge at (0,0). This is why this bifusation is called as transcritical hifurcation. The Jacobian matrix of the system (19.15) is given by

$$J(x,y) = \begin{pmatrix} \frac{\alpha}{d} & \frac{\alpha}{k} \\ \frac{a}{d} & \frac{\alpha}{b} \end{pmatrix} = \begin{pmatrix} \mu - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

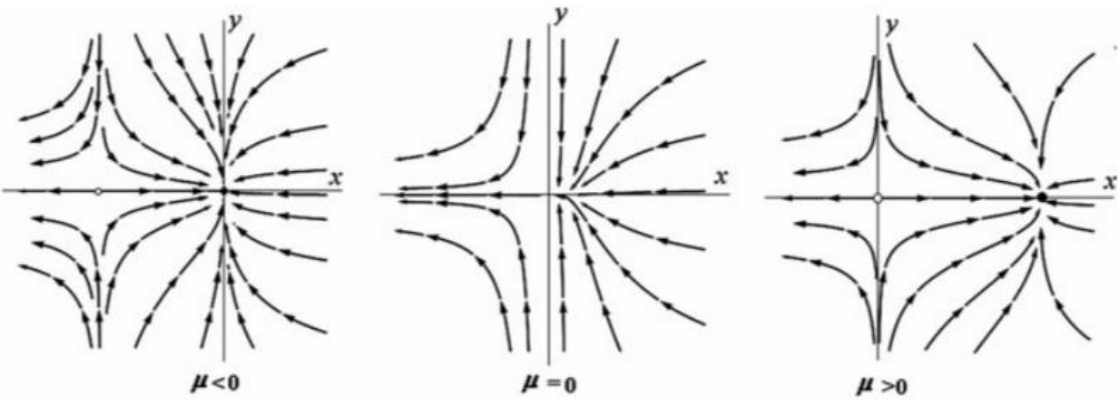
At the poimt (0, 0),

$$J(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}$$

which has eigenvalues μ and (-1). Therefore, the fixed point (0,0) of the system (19.15) is a stable node if $\mu < 0$ and it is a saddle point if $\mu > 0$. For $\mu = 0$, the fixed point is atmi-stable. Agrin, at $(\mu, 0)$,

$$J(\mu, 0) = \begin{pmatrix} -\mu & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvalues of $J(\mu, 0)$ are $(-\mu)$ and (-1), showing that the fixed point $(\mu, 0)$ is a stable node if $\mu > 0$, and a saddle pcint if $\mu < 0$. The phase diagrams for different signs of μ are shown in Fig.



From the diagram, we see that the behaviour of the system changes when the panmekerμ passes through the ofigin. In this stage, the saddle becomes a stablenode and the stable node becomes a saddle. That is, when μ passes through the origin from left, the fixed point origin changes to a saddle from a stable node and the fixed point $(\mu, 0)$ changes from a saddle to a stable node. This type of bifurcation is known as transcritical bifurcation. Here $\mu = 0$ is the bifurcation point. The feature is same as in one-dimensional system where no fixed points are disappeared.

19.4.3 Pitchfork Bifurcation;

There are two types of pitchfork bifurcations, namely supercritical and subcritical pitchfork bifurcations. In the present section, we deal with these two bifurcations scenario, first the supercritical pitchfork bifurcation and then the subcritical pitchfork bifurcation will be illustrated.

Consider a two-dimensional system represented by

$$\dot{x} = \mu x - x^3, \dot{y} = -y; \qquad (19.16)$$

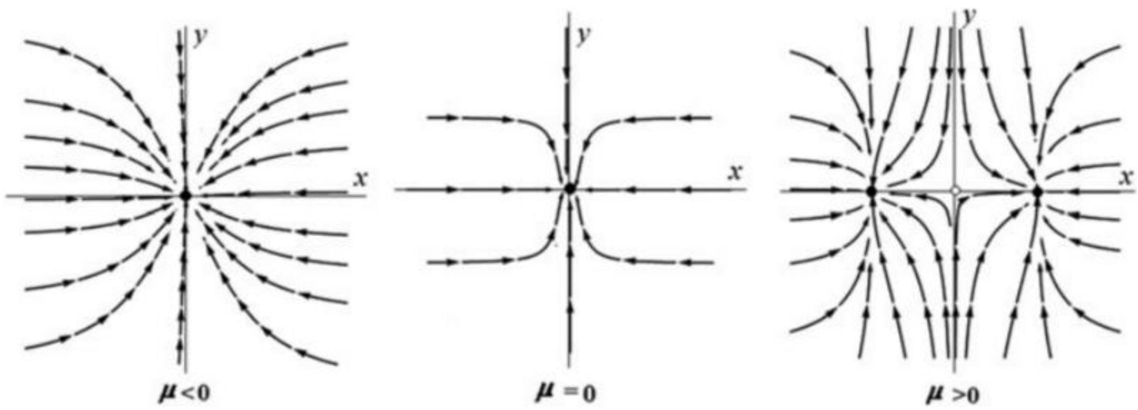
where $\mu \in \mathbb{R}$ is the parameter. For $\mu < 0$, the system (19.16) has only one equilibrium point at the origin. The Jacobian matrix at this fixed point is given by

$$J(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of $J(0,0)$ are $\mu, (-1)$, showing that the fixed point ongin is a stable node. For $\mu > 0$, the system has three fixed points $(0,0), (\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$. The Jacobian matrix of (19.16) calculated at these fixed points are given by

$$J(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}, J(\sqrt{\mu}, 0) = \begin{pmatrix} -2\mu & 0 \\ 0 & -1 \end{pmatrix}, J(-\sqrt{\mu}, 0) = \begin{pmatrix} -2\mu & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of $J(0,0)$ are $\mu, (-1)$, which are opposite in signs. So, the equilibri um point $(0,0)$ is a saddle for $\mu > 0$. Clearly, the eigenvalues of Jacobian matrix show that the other two fixed points are stable nodes. The phase diagrams for different values of the bifurcation parameter μ are presented in Fig.



The diagram shows that as soon as the parameter μ crosses the bifurcation point origin, the fixed point origin bifurcates into a saddle point from a stable node. In this situation, it also gives birth to two stable nodes at the points $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$. The amplitudes of the newly created stable nodes grow with the parameter. This type of bifurcation is known as supercritical pitchfork bifurcation. We shall now discuss the subcritical pitchfork bifurcation. Consider a parameter-dependent two-dimensional system represented by

$x = \mu x + x^3, y = -5$ (19.17)

with the parameter $\mu \in \mathbf{R}$. When $\mu < 0$, the system (19.17) has three distinctflxed point, namely (0,0), $(\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$. The Jacobians of the system evaluated at these fixed points are given by

$$J(0,0) - \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}, J(\sqrt{\mu}, 0) - \begin{pmatrix} 4\mu & 0 \\ 0 & -1 \end{pmatrix}, J(-\sqrt{\mu}, 0) - \begin{pmatrix} 4\mu & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of $J(0,0)$ are $\mu, (-1)$, which are of same sign. Thus, the fixed point origin is a stable node for $\mu < 0$. Similarly, calculating the eigenvalues of the other two Jacobian matrices of the system one can see that the fixed points $(\pm\sqrt{\mu}, 0)$ are saddle points. For $\mu > 0$, the system has a single fixed point at the origin, which is saddle. If we draw the phase portrait of the system, then we can see that as soon as the parameter crosses the bifurcation point $\mu = 0$, the stable node at the origin coincides with the sadiles and then bifurcates into a saddle. This type of bifurcation is known as subcritical pitchfork bifurcation.

19.4.4 Hopf Bifurcation:

So far we have discussed bifurcations of systems with real eigenvalues, either positive or negative, of the corresponding Jacobian matrix evaluated at the fixed points of the corresponding system. We shall now discuss a very interesting periodic bifurcation phenomenon for a two-dimensional system where the eigenvalues are complex. This type of bifurcating phenomenon in two-dimensional or higher dimensional systems was studied by the German Scientist EherhardHopf (1902-1983) and it was named Hopf bifurcation due to the recognition of his work. This type of bifurcation was also recognized by Henri Poincaré and later by A.D. Andronov in 1930. Hopf bifurcation occurs when a stable equilibrium point lossesits stability and gives birth to a limit cycle and vice versa. There are two types of Hopf bifurcations, viz., supercritical and subcritical Hopf bifurcations. When stable limit cycles are created for an unstable equilibrium point, then the bifurcation is called a supercritical Hopf bifurcation. In engineering applications point of view, this type of bifurcation is also termed as soft or safe bifurcation because the amplitude of the limit cycles build up gradually as the parameter varies from the bifurcation point. On the other hand, when an unstable limit cycle is created for a stable equilibrium point, then the bifurcation is called a subcritical Hopf bifurcation. It is also known as a hard bifurcation. In case of subcritical Hopf bifurcation, a steady state solution could bocome unstable as parameter varies and the nonzero solutions coukd tend to infinity. We shall now illustrate the supercritical and subcritical Hopf bifurcations below.

19.4.4.1 Supercritical Hopf Bifurcation:

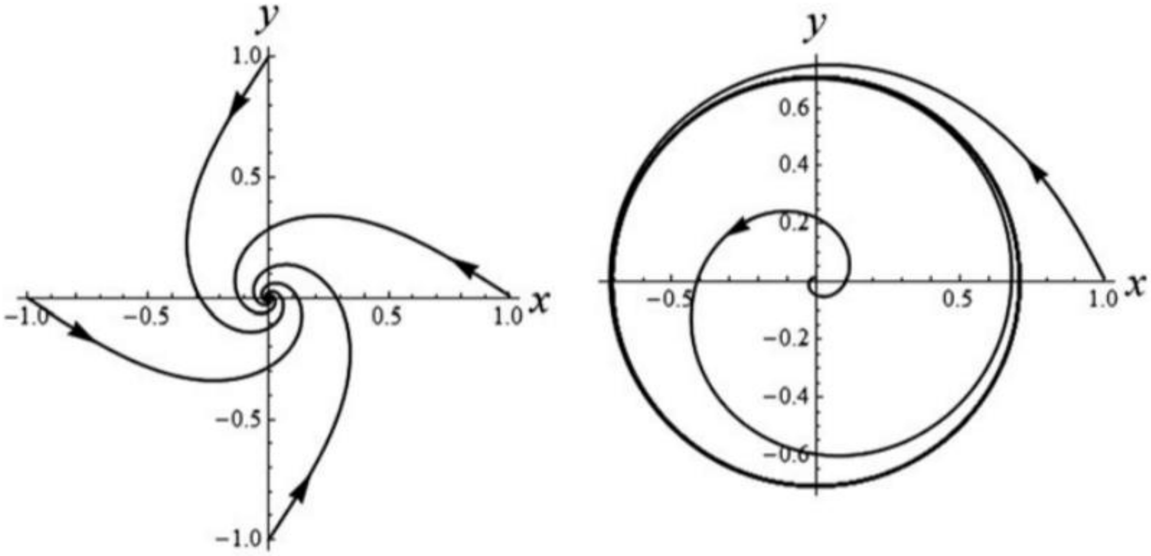
Consider a two-dimensional system with parameter $\mu \in \mathbb{R}$,

$\dot{x} = \mu x - y - x(x^2 + y^2), \dot{y} = x + \mu y - y(x^2 + y^2).$ (19.18)

The system has a unique fixod point at the origin. In polar coordinates, the system can be written as

$$\dot{r} = \mu r - r^3, \dot{\theta} = 1,$$

which are decoupled, and so easy to analyze. The phase portraits for $\mu < 0$ and $\mu > 0$ are shown in Fig.



μ =-0.5 μ =0.5

When $\mu < 0$, the fixed point origin $(r = 0)$ is a stable spiral and all trajectories are attracted to it in anti-clockwise direction. For $\mu = 0$, the origin is still a stable spiral, though very weak. For $\mu > 0$, the origin is an unstable spiral, and in this case, there is a stable limit cycle at $r = \sqrt{\mu}$. We are now interested to see how the eigenvalues behave when the parameter is varying. The Jacobian matrix at the fixed point origin is calculated as

$$J(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

which has the eigenvalues $(\mu \pm i)$. Thus origin is a stable spiral when $\mu < 0$ and an unstable spiral when $\mu > 0$. Therefore as expected the eigenvalues cross the imaginary axis from left to right as the parameter changes from negative to positive values. Thus we see that a supercritical Hopf bifurcation occurs when a stable spiral changes into an unstable spiral surrounded by a limit cycle.

19.4.4.2 Subcritical Hopf Bifurcation:

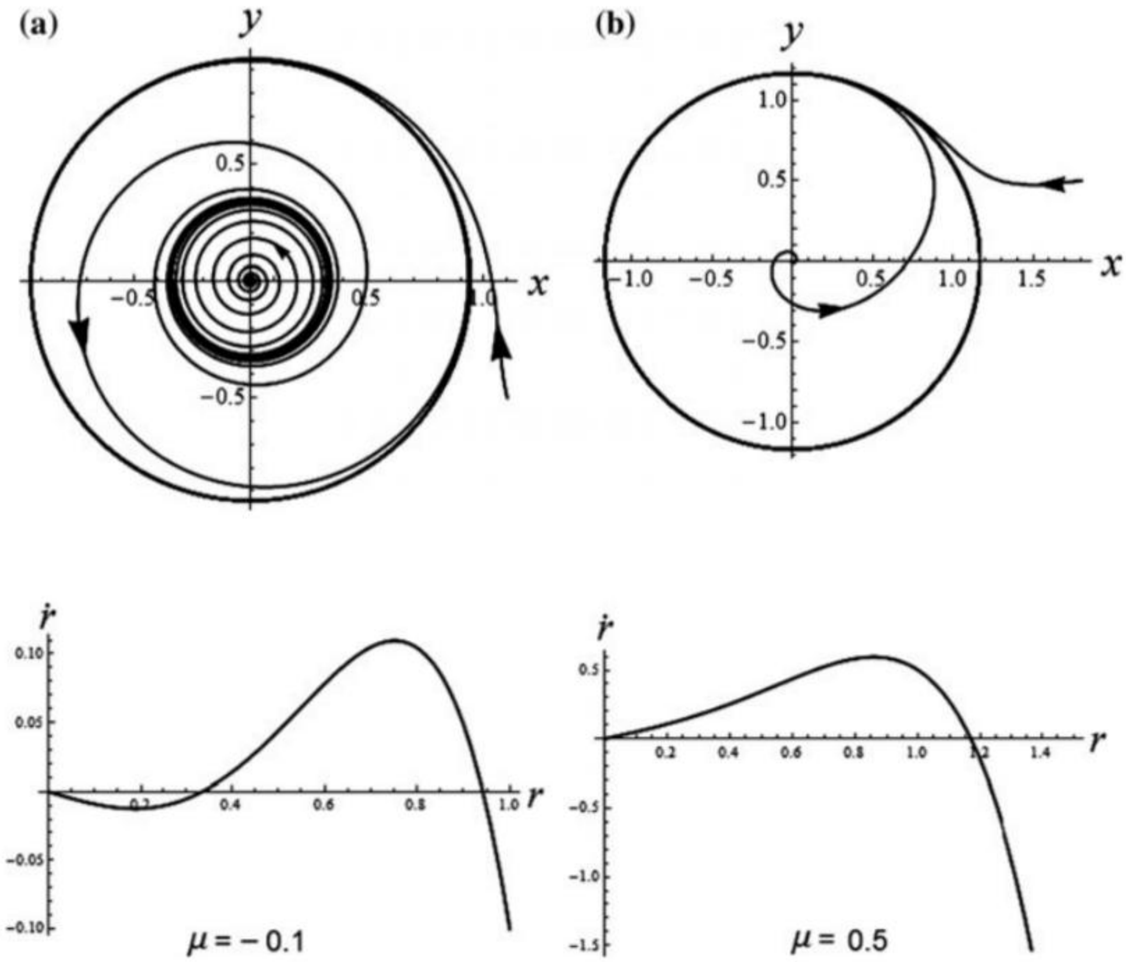
Consider a two-dimensional system represented by

$$\begin{aligned} \dot{x} &= \mu x - y + x(x^2 + y^2) - x(x^2 + y^2)^2 \\ \dot{y} &= x + \mu y + y(x^2 + y^2) - y(x^2 + y^2)^2 \end{aligned} \qquad (19.19)$$

where $\mu \in \mathbb{R}$ is the parameter. In polar coordinates, the system can be transformed as

$$\dot{r} = \mu r + r^3 - r^5, \dot{\theta} = 1.$$

This system has a unique fixed point at the origin. The phase portraits for $\mu < 0$ and $\mu > 0$ are presented in Fig.



From the phase diagram it is clear that when $\mu > 0$, the fixed point origin $(r = 0)$ is a stable spiral and all trajectories are attracted to it in anti-clockwise direction, and for $\mu < 0$, it is an unstable spiral. The diagram also exhibits that the system has two limits cycles when $\mu < 0$, one of which is stable and other is unstable. For $\mu > 0$, it has only a stable limit cycle. All these cycles can be determined from the equation $\mu + r^2 - r^4 = 0$. For $\mu < 0$, the system (19.19) has two limit cycles at

$$r^2 = \frac{1 \pm \sqrt{1 + 4\mu}}{2},$$

and for $\mu > 0$, the unique limit cycle occurs at

$$r^2 = \frac{1+\sqrt{1+4\mu}}{2}.$$

A sketch of $\dot{r}$ versus $(\mu r + r^3 - r^5)$ for two different values of μ is shown in Fig.

From this figure it is clear that when $\mu < 0$, the limit cycle at $r^2 = \frac{1+\sqrt{1+4x}}{2}$ is stable, while the limit cycle at $r^2 = \frac{1-\sqrt{1+2\overline{8}}}{2}$ is unstable, and for $\mu > 0$, the limit cycle at $r^2 = \frac{1+\sqrt{1+4\bar{x}}}{2}$ is stable.

Theorem 19.3 (Hopf bifurcation) Let (x_0, y_0) be an equilibrium point of a planer autonomous system

$$\dot{x} = f(x, y, \mu), \quad \dot{y} = g(x, y, \mu)$$

depending on some parameter $\mu \in \mathbb{R}$, and let

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial_g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix},$$

the Jacobian matrix of the system evaluated at the equilibrium point has purely imaginary eigenvalues $\lambda + (\mu) = i\omega, \lambda - (\mu) = -i\omega; \omega \neq 0$ at $\mu = \mu_0$. If

(i) $\frac{\mathrm{d}}{\mathrm{d}}(\mathrm{Re}\,\lambda(\mu)) = \frac{1}{2} > 0$ at $\mu = 0$,

(ii) $(f_{\mu x} + g_{\mu y}) = 1 \neq 0$, and

(iii) $a = -\frac{\mu}{8} \neq 0$ for $\mu \neq 0$.

where the constant a is given by

$$a = \frac{1}{16}(f_{xx} + g_{xxy} + f_{xyy} + g_{yyy}) + \frac{1}{160}\{f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\},$$

evaluated at the equilibrium point, then a periodic solution bifurcates from the equilibrium point (x_0, y_0) into $\mu < \mu_0$ if $a(f_{\mu x} + g_{\mu}) > 0$ or into $\mu > \mu_0$ if $a(f_{\mu x} + g_{\mu y}) < 0$ Also, the equilibrium point is stable for $\mu > \mu_0$ (respectively $\mu < \mu_0$) and unstable for $\mu < \mu_0$ (respectively $\mu > \mu_0$) if $(f_{\mu x} + g_{\mu \mathrm{f}}) < 0$ (respectively > 0). In both the cases, the periodic solution is stable (respectively unstable) if the equilibrium point is unstable (respectively stable) on the side of $\mu = \mu_0$ for which the periodic solutions exist.

We now illustrate the Hopf bifurcation theorem by considering the well-known van der Pol oscillator. The equation for van der Pol is given by $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \mu \geq 0$. Setting $\dot{x} = y$, the equation can be written as

$$\left.\begin{aligned} \dot{x} &= y = f(x, y, \mu) \\ \dot{y} &= -x + \mu(1 - x^2)y = g(x, y, \mu) \end{aligned}\right\}$$

The system has the equilibrium point $(x_0, y_0) = (0,0)$, and the corresponding Jacobian matrix at $(0,0)$ has the eigenvalues

$$\lambda_{\pm}(\mu) = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2},$$

which are complex for $0 \leq \mu < 2$ and real for $\mu \geq 2$. For $\mu = 0$, the eigenvalues are purely imaginary: $\lambda_+(\mu) = i, \lambda_-(\mu) = -i$. Now, at the equilibrium point

(i) $\frac{\mathrm{d}}{\mathrm{d}\mu}(\mathrm{Re}\,\lambda(\mu)) = \frac{1}{2} > 0$ at $\mu = 0$,

(ii) $\left(f_{\mu x} + g_{\mu y}\right) = 1 \neq 0$, and

(iii) $a = -\frac{\mu}{8} \neq 0$ for $\mu \neq 0$.

Also, at this point, we see that $a\left(f_{\mu x} + g_{\mu y}\right) = -\frac{\mu}{8} < 0$ for $\mu > 0$. So, by Theorem 19.3, the system has a periodic solution (limit cycle) for $\mu > 0$. The stability of the limit cycle depends on the sign of $\left(f_{\mu v} + g_{\mu q}\right)$, which is positive (equal to 1) at (0,0). Hence for $\mu > 0$ the equilibrium point origin must be unstable and the limit cycle must be stable.

Exercises:

1. Determine the bifurcation points for the system $\dot{x} = \mu - A\cos(\pi x)$. Sketch the flow in the (x, u) plane.
2. Draw the bifurcation diagram for the following systems when the parameter $\mu \in \mathbb{R}$ varies:
   (i) $\dot{r} = r(\mu + r), \dot{\theta} = -1$
   (ii) $\dot{r} = r(\mu - r)(\mu - r^2), \dot{\theta} = -1$
3. Discuss Hopf bifurcation for the system $\dot{r} = r(\mu - r^2), \dot{\theta} = -1; \mu \in \mathbb{R}$.
4. Plot phase portraits and also sketch the bifurcation diagrams for the following systems
   (i) $\dot{x} = x, \dot{y} = \mu - y^4, \mu \in \mathbb{R}$

UNIT-20

Randomness of orbits of a dynamical system: The Lorentz equations, Chaos, Strange attractors.

20.1 Introduction:

A non-linear system can have a more complicated steady-state behaviour that is not equilibrium, periodic oscillation, or almost periodic oscillation. Such behaviour is referred to as chaos. Such of these chaotic motions exhibit randomness, despite the deterministic nature of the system.

In this chapter we shall sketch a number of complicated phenomena which are tied to the concept of chaos and strange attractive we shall restrict ourselves to autonomous differential equation with dimension n≥ 3.

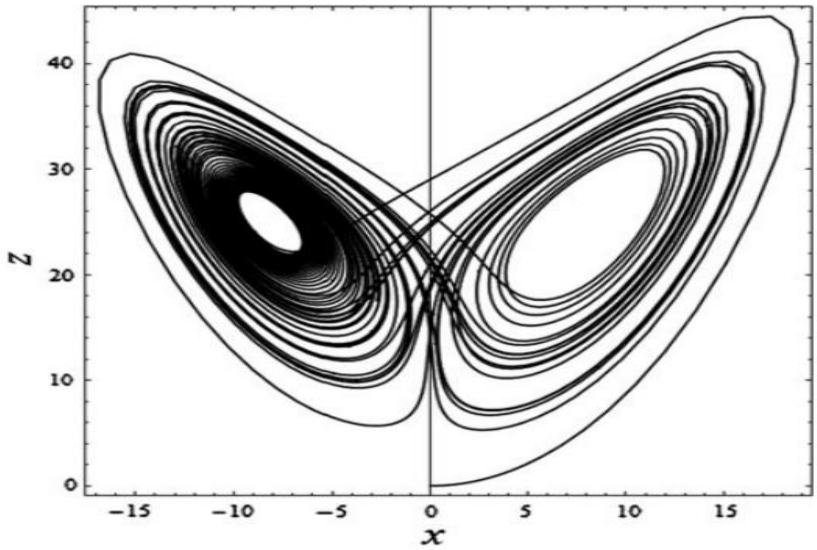
20.2: The Lorentz Equation:

The Massachusetts Institute of Technology (M.I.T.) Meteorologist Edward Norton Lorenz (1917-2008) in the year 1963 had derived a three-dimensional system from a drastically simplified model of convection rolls in atmospheric flow. The simplified model may be written in normalized form as follows:

$$\left.\begin{array}{l}\dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz\end{array}\right\} \qquad (20.1)$$

where $\sigma, r, b > 0$ are all parameters. The system (20.1) has two simple nonlinear terms xz and xy in the second and third equations, respectively. Lorenz discovered that this simple looking deterministic system could have extremely erratic or complicated dynamics over a wide range of parameter values σ, r, and b. The solutions oscillate irregularly in a bounded phase space. When he plotted the trajectories in three dimensions, he discovered a new concept in the theory of dynamical system. Moreover, unlike stable fixed points or limit cycle, the strange attractor appeared in the phase space is not a point neither a curve nor a surface. It is a fractal with fractional dimension between 2 and 3 . We shall study this simple looking system thoroughly below.

Consider a fluid layer of depth h, confined between two very long, stress-free, rigid and isothermal, horizontal plates in which the lower plate has a temperature T_0 and the upper plate has a temperature T_1 with $T_0 > T_1$. Let $\Delta T = T_0 - T_1$ be the temperature difference between the plates. As long as the control parameter the temperature difference ΔT is small, the fluid layer remains static and so it is stable. As ΔT crosses a critical value, this static fluid layer becomes unstable and as a result a convection roll appears in the fluid layer. This phenomenon is known as thermal convection. We take the x-axis in the horizontal direction and the z-axis in the vertical direction. From the symmetry of the problem, all flow variables are independent of the y-coordinate and the velocity of the fluid in the y-direction is zero. Under Boussinesq approximations (the effects of temperature is considered only for body force term in the equation of motion), the governing equations of motions for incompressible fluid flows viz., the continuity equation, momentum equations, and thermal convection may be written for usual notations as (Batchelor, Chandrasekhar)



$$\nabla \cdot V = 0 \qquad (20.2)$$

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)W = -\frac{1}{\rho_0}\nabla p + v\nabla^2 V - \frac{\rho}{\rho_0} g\bar{z} \qquad (20.3)$$

$$\frac{\partial T}{\partial t} + (\bar{v} \cdot \nabla)T = x\nabla^2 T \qquad (20.4)$$

where $\rho = \rho(T)$ is the fluid densily at vemperature T as given by

$$\rho(T) = \rho_0\{1 - a(T - T_0)\} \qquad (20.5)$$

$\rho_0 = \rho(T_0)$ is the fluid density at the reference tempenture $T_{0.}, (= \mu/\rho)$ the kinematic viscosity of the fluid, μ heing the coefficiem of dynamic fluid viscosity, α the coefficient of thermal expansion, x the coefficient of thermal expansion, g the acceleration of gravity acting in the downward direction, z the wit vector aldng the z axis, $v = (u, 0, w)$ is the fluid vekcily at some instant t in the convectional mootion, and $T = T(x, z, t)$ is the temperatus of the fluid at that tirse.

The boundary conditions are prescribed as follows:

$$T = T_0 \text{ at } z = 0 \text{ and } T = T_1 \text{ at } z = h.$$

Consider the perturbed quantities (when convection starts) T', p' and p' defined as

$$T = T_b(z) + T^v(x, z_1 t), p = p_b(z) + p^{'}(x, z_1 t) \text{ and } \rho = p_b(z) + \rho(x, z_1 t).$$

where $T_b(z) = T_0 - (T_0 - T_1)\&$ is the temperature at the skady state, $\rho_b(z) = p_0\{1 - \alpha(T_b(z) - T_{01})\}$ is the comreipending fluid density, and $p_b(z)$ is the comresponding pressure given by $q_b/dz = -gp_b(z)$, which is chatined in the conduction stare and by putting $\boldsymbol{y} = 0$ in the eqution of motion (20.3).
Sulstituting these in the Fq. (20.3)-(20.5), we get

$$\begin{aligned} &\frac{\partial T^{'}}{\partial t} + (V - \nabla)T^{'} - \frac{(T_0 - T_1)}{h} = k\nabla^2 T^t \\ &p^{'} = -p_0 \alpha T^{'} \end{aligned} \qquad (20.6)$$

We have

$$\frac{\partial V}{\partial t} + (V, \nabla)\underline{V} = -\frac{1}{\rho_0}\nabla p^{'} + v\nabla^2 V - \alpha T^{'} z \qquad (20.7)$$

The boundary conditions become

$$T^{'} = 0 \text{ at } z = 0, h \qquad (20.8)$$

Consider the dimensionless quantities

$$x^* = \frac{x}{h}, z^* = \frac{z}{h}, \underset{\sim}{*}^* = \frac{h}{k}\underset{\sim}{2}, t^* = \frac{k}{h^2}t, p^* = \frac{h^2}{k^2}p^{'}, \theta = \frac{T^{'}}{T_0 - T_1}$$

where θ° represents the temperature deviation. Then Eqs. (20.2), (20.6), and (20.7), respectively, become (omitting the asterisk (*) for the dimensionless quantities)

$$\begin{aligned} &\nabla \cdot \underset{\sim}{V} = 0 \\ &\frac{\partial \underset{\sim}{\partial t}}{\underset{\sim}{I}} + (\underset{\sim}{V} \cdot \nabla)\underset{\sim}{V} = -\frac{1}{\rho_0}\nabla p + \sigma\nabla^2 \underset{\sim}{V} - \sigma R \theta \bar{z} \\ &\frac{\partial \theta}{\partial t} + (\underset{\sim}{V} \cdot \nabla)\theta = \nabla^2 \theta \end{aligned} \qquad (20.9)$$

where $\sigma = v/\kappa$ is the Prandtl number measuring the ratio of fluid kinematic viscosity and the thermal diffusivity and $R = \alpha g(T_0 - T_1)h^3/v\kappa$ is the Rayleigh number characterizing basically the ratio of temperature gradient and the product of the kinematic fluid viscosity and the thermal diffusivity. Again, the boundary conditions become

$$\theta = 0 \text{ at } z = 0,1. \qquad (20.10)$$

This is known as Rayleigh-Benard convection in the literature. Let $\psi = \psi(x, z, t)$ be the steam function which is a scalar function representing a curve in the fluid medium in which tangent at each point gives velocity vector and satisfying the following relations for two-dimensional flow consideration

$$u = -\frac{\partial \psi}{\partial z}, w = \frac{\partial \psi}{\partial x}. \tag{20.11}$$

Then the continuity equation is automatically satisfied. Also, in this case, the vorticity vector has only one nonzero component ω in the y-direction expressed by

$$\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x^2} = -\nabla^2 \psi. \tag{20.12}$$

Taking curl of the 2nd Eq. of (20.9) and then projecting the modified equation in the y-direction, we have

$$\frac{\partial \omega}{\partial t} + (\underset{\sim}{V} \cdot \nabla)\omega = \sigma \nabla^2 \omega - \sigma R \frac{\partial \theta}{\partial x} \tag{20.13}$$

But,

$$\begin{aligned}(\underset{\sim}{V} \cdot \nabla)\omega &= u\frac{\partial \omega}{\partial x} + w\frac{\partial \omega}{\partial z} \\ &= -\frac{\partial \psi}{\partial z}\frac{\partial \omega}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial z} \\ &= \frac{\partial(\omega, \psi)}{\partial(x, z)} \\ &= J(\omega, \psi)\end{aligned}$$

Similarly, $(\underset{\sim}{V}, \nabla)\theta = J(\theta, \psi)$. Therefore, the Eqs. (20.9) and (20.13), respectively, reduce to

$$\begin{aligned}\frac{\partial \theta}{\partial t} + J(\theta, \psi) - w = \nabla^2 \theta \\ \frac{\partial \omega}{\partial t} + J(\omega, \psi) = \sigma \nabla^2 \omega - \sigma R \frac{\partial \theta}{\partial x}\end{aligned} \tag{20.14}$$

where

$$\omega = -\nabla^2 \psi. \tag{20.15}$$

We assumed that the boundaries $z = 0,1$ are stress-free, an idealized boundary conditions. So we have other boundary conditions as given by

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = 0 \text{ at } z = 0,1. \tag{20.16}$$

We shall now convert the above set PDEs (20.14) into ODEs using Galerkin expansion of ψ and θ. Let the Galerkin expansions of ψ and θ satisfying the boundary conditions be

$$\begin{aligned}\psi(x, z, t) = A(t)\sin(\pi z)\sin(kx). \\ \theta(x, z, t) = B(t)\sin(\pi z)\cos(kx) - C(t)\sin(2\pi z).\end{aligned} \tag{20.17}$$

where k is the wave number and $A(t), B(t)$, and $C(t)$ are some functions of time t. Then $\omega = -\nabla^2 \psi = (\pi^2 + k^2)\psi, \nabla^2 \omega = -\nabla^2 \psi = -(\pi^2 + k^2)^2 \psi$ and $J(\omega, \psi) = 0$.

Therefore, the 2[nd]Eq. of (20.14) gives

$$(\pi^2 + k^2)\frac{\mathrm{d}A}{\mathrm{d}t}\sin(\pi z)\sin(kx) = \quad kR\sigma B(t)\sin(\pi z)\sin(kx) \\ -\sigma(\pi^2 + k^2)^2 A(t)\sin(\pi z)\sin(kx).$$

This is true for all values of x and z. Therefore, we must have

$$\begin{aligned}(\pi^2 + k^2)\frac{\mathrm{d}A}{\mathrm{d}t} &= kR\sigma B(t) - \sigma(\pi^2 + k^2)^2 A(t) \\ \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{kR\sigma}{(\pi^2+k^2)}B(t) - \sigma(\pi^2 + k^2)A(t)\end{aligned} \qquad (20.18)$$

Similarly, from the Eq. (20.13), we get

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} &= kA(t) - (\pi^2 + k^2)B(t) - \pi k A(t)C(t) \\ \frac{\mathrm{d}C}{\mathrm{d}t} &= \frac{\pi k}{2}A(t)B(t) - 4\pi^2 C(t)\end{aligned} \qquad (20.19)$$

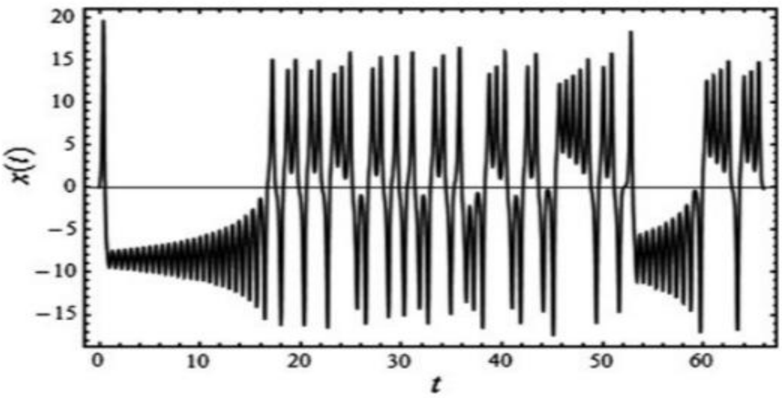
Rescale the variables $t, A(t), B(t)$, and $C(t)$ as follows:

$$\begin{aligned}\tau &= (\pi^2 + k^2)t, X(\tau) = \frac{k/k_c}{2 + (k/k_c)^2}A(t), \\ Y(\tau) &= \frac{(k/k_c)^2 R}{k_c^3\{2 + (k/k_c)^2\}^3}B(t) \text{ and } Z(\tau) = \frac{\sqrt{2}(k/k_c)^2 R}{k_c^3\{2 + (k/k_c)^2\}^3}C(t),\end{aligned}$$

where $k_c = \frac{\pi}{\sqrt{2}}$ is the wave number corresponding to the convection threshold. Substituting these in the Eqs. (20.18)-(20.19), we finally obtain the Lorenz equations as

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \sigma(Y - X) \\ \frac{\mathrm{d}y}{\mathrm{d}} &= rX - Y - XZ \\ \frac{\mathrm{d}}{\mathrm{d}\tau} &= XY - bZ\end{aligned}\right\} \qquad (20.20)$$

where $r = R/R_c$ is known as the reduced Rayleigh number, $b = 8/(2 + (k/k_c)^2)$, and $R_c = (\pi^2 + k^2)^3/k^2$. Using the wave number corresponding to the convection threshold, that is, using $k = k_c$, we get $R_c = 27\pi^4/4$ and $b = 8/3$. The system (6.54) is an autonomous system of dimension three. The system, although looks very simple, is very complicated to solve analytically, because the system represents a set of nonlinear equations in $\mathbb{R}^3$ with the nonlinear terms XZ and XY in the



In considering the phenomenon of connective in the atmosphere of the earth by hitting from below and cooling from above, Lorentz derived the following system of non-linear equations;-

$$\dot{x} = \sigma(y - x),$$

$\dot{y} = rx - y - xz,$ …………….(20.21)

$$\dot{z} = xy - bz$$

In which σ,r and b are the parameters in the sequel, we shall taken-

$\sigma = 10, b = \frac{8}{3},$ and r can taken varies +ve values. First we taken $r = 28$ and connect a numerical approximation of a solution which start in a nbd of the unstable equilibrium solution; If starts in E_u, 20 that the orbits follows the unstable manifold W_u .

From the above orbit diagram for Lorenz system, the following qualitative features can be drawn:

(a) The orbit is not closed;

(b) The orbit diagram or the set of trajectories do not depict a transition stage but a well-organized regular structure;

(c) The orbit describes a number of loops on the left and on the right without any regularity in he number of loops and the loops on both sides are in opposite directions of rotations;

(d) The number of loops on the left and on the right depends in a very sensitive way on the infinitesimal change of initial conditions. Transient solution does not exhibit any periodic pattern.

(e) This is an attracting set with a dimension greater than two and was named “strange attractor” by Ruelle and Takens.

A number of characteristic of the Lorentz equations easily derived. We discuss them briefly. For this we need the following lemma which we state without proof.

Lemma 20.1:

Consider the equation $\dot{\vec{x}} = \vec{f}(\vec{x})$ in R^n and a domain D(0) in R^n which is supposed to have a value $v(0)$. The flow defines a mapping g of D(0) in R^n, $g: R^n \to R^n, D(t) = g^t D(0)$. For the value $v(t)$ of the domain D(t), we have

$$\left.\frac{dv}{dt}\right|_{t=0} = \int_{D(0)} \vec{\nabla} \cdot \vec{f} \, dv.$$

The characteristic are

(i) The equation (20.21) have the following reflect symmetry:

If replace $x, y, z$by $-x, -y, +z$ then the equations have the same form. It follows that each solution $x(t), y(t), z(t)$ has a symmetric counterpart $(-x(t), -y(t), +z(t))$ which is also a solution.

(ii) z-axis of the Lorenz system is invariant.
If we initially take $x = y = 0$ in the Lorenz system (20.21), we see that $x = y = 0$ for all future time t. In this case, the system gives

$$z = -bz \Rightarrow z(t) = z(0)e^{-bt} \to 0 \text{ as } t \to \infty.$$

Therefore the z-axis, that is, $x = y = 0$ is an invariant set and all solutions starting on the z-axis will tend to the origin $(0,0,0)$ as $t \to \infty$.
(iii) Lorenz system is dissipative in nature.
In the dissipative system, the volume occupied in the phase space decreases as the system evolves in time. Let $V(t)$ be an arbitrary volume enclosed by a closed surface $S(t)$ in the phase space and let $S(t)$ changes to $S(t + dt)$ in the time interval dt. Let $\hat{n}$ be the outward drawn unit normal to the surface S. f is the velocity of any point, then the dot product $(\underset{\sim}{f} \cdot \hat{n})$ is the outward normal component of velocity.
Therefore, in time dt, a small elementary area dA sweeps out a volume $(\underset{\sim}{f} \cdot \hat{n})\mathrm{d}Adt$.
Therefore, $V(t + dt) = V(t) +$ (volume swept out by small area of surface which is integrated over all such elementary areas).
Hence we get

$$V(t + \mathrm{d}t) = V(t) + \iint_S (\underset{\sim}{f} \cdot \hat{n})\mathrm{d}A\mathrm{d}t$$

$$\Rightarrow \frac{V(t + \mathrm{d}t) - V(t)}{\mathrm{d}t} = \iint_s (\underset{\sim}{f} \cdot \hat{n})\mathrm{d}A = \int \quad (\nabla \cdot f)\mathrm{d}V [\text{ Divergence Theorem}]$$

$$\Rightarrow \dot{V}(t) = \frac{\mathrm{d}V}{\mathrm{d}t} = \int_V \left(\nabla \cdot \underset{\sim}{f}\right)\mathrm{d}V \qquad (20.22)$$

So for the Lorenz system, we have

$$\begin{aligned}\nabla \cdot \underset{\sim}{f} &= \frac{\partial}{\partial x}(\sigma(y-x)) + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz) \\ &= -(\sigma + 1 + b).\end{aligned}$$

Therefore, from (20.22)

$$\dot{V} = \int_V -(\sigma + 1 + b)dV = -(\sigma + 1 + b)V$$

which gives the solution $V(t) = V(0)e^{-(1+a+b)t}, V(0)$ being the initial volume. This implies that the volumes in the phase space decreases (shrink) exponentially fast and finally reaches an attracting set of zero volume. Hence, Lorenz system is dissipative in nature.
(iv) Lorenz system shows a pitchfork bifurcation at origin when $r \to 1$.
The fixed points of the Lorenz system are obtained by solving the equations

$$\sigma(y - x) = 0, rx - y - xz = 0, xy - bz = 0.$$

These give

$$x = y = z = 0 \text{ and } x = y = \pm\sqrt{b(r-1)}, z = (r-1).$$

Clearly, the origin $(0,0,0)$ is a fixed point for all values of the parameters. The system has another two fixed points for $r > 1$, which are given by

$$x^* = y^* = \pm\sqrt{b(r-1)}, z^* = (r-1).$$

Lorenz called these fixed points as

$$\begin{aligned}c^+ &= (\sqrt{b(r-1)}, \sqrt{b(r-1)}, (r-1)) \text{ and} \\ c^- &= (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, (r-1)).\end{aligned}$$

Clearly, these two fixed points are symmetric in x and y coordinates. As $r \to 1$, they coincide with the fixed point origin, which gives a pitchfork bifurcation of the system. The fixed point origin is the bifurcating point. It is impossible for the Lorenz system to have either repelling fixed points or repelling closed orbits.

(v) Linear stability analysis of the Lorenz system about the fixed point origin The linearized form of the Lorerz system about the fixed point origin is given by

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y \\ \dot{z} &= -bz\end{aligned}$$

Now, the z-equation is decoupled so it gives

$$\dot{z} = -bz \Rightarrow z(t) = z(0)e^{-bt} \to 0 \text{ as } t \to \infty.$$

The other two equations can be written as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}-\sigma & \sigma\\ r & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Hence sum of the diagonal elements of the matrix, $\tau=-\sigma-1=-(\sigma+1)<0$ and its determinant $\Delta=(-\sigma)(-1)-\sigma r=\sigma(1-r)$. If $r>1$, then $\Delta<0$ and so the fixed point origin is a saddle. Since the system is three dimensional, a new type of saddle is created. This saddle has one outgoing and two incoming directions. If $r<1$, then $\Delta>0$ and all directions are incoming and the fixed point origin is a sink (stable node).
(vi) The fuxed point origin of the Lorenz system is globally stable for $0<r<1$. Let us consider the Lyapunov function for the Lorenz system as

$$V(x,y,z)=\frac{x^2}{\sigma}+y^2+z^2.$$

Then the directional derivative or orbital derivative is given by

$$\begin{aligned}\dot{V}&=\frac{2x\dot{x}}{\sigma}+2y\dot{y}+2z\dot{z}\\ \Rightarrow\frac{\dot{V}}{2}&=\frac{x\dot{x}}{\sigma}+y\dot{y}+z\dot{z}\\ &=x(y-x)+y(rx-y-xz)+z(xy-bz)\\ &=-x^2+(1+r)xy-y^2-bz^2\\ &=-\left[x^2-2x\left(\frac{1+r}{2}\right)xy+\left(\frac{1+r}{2}\right)^2y^2\right]+\left(\frac{1+r}{2}\right)^2y^2-y^2-bz^2\\ &=-\left[x-\left(\frac{1+r}{2}\right)y\right]^2-\left[1-\left(\frac{1+r}{2}\right)^2\right]y^2-bz^2.\end{aligned}$$

Thus we see that $\dot{V}<0$ if $r<1$ for all $(x,y,z)\neq(0,0,0)$ and $\dot{V}=0$ iff $(x,y,z)=(0,0,0)$. Therefore, according to Lyapunov stability theorem, the fixed point origin of the Lorenz system is globally stable if the parameter $r<1$.
(vii) Linear stability at the fixed points $c^{\pm}$.
Eigenvalues of the Jacobian matrix at the critical points $c^{\pm}$of the Lorenz system satisfy the equation

$$\lambda^3+(\sigma+b+1)\lambda^2+b(\sigma+r)\lambda+2\sigma b(r-1)=0.$$

For $1<r<r_H$, the three roots of the above cubic equation have all negative real parts, where

$$r_H=\frac{\sigma(3+b+\sigma)}{\sigma-b-1}.$$

If $r=r_H$, two of the eigenvalues are purely imaginary, and so Hopf bifurcation occurs. This bifurcation turns out to be subcritical for $r<r_H$, where two unstable periodic solutions exist

for two critical values of fixed points. At $r = r_H$ these periodic solutions disappeared. For $r > r_H$ each of the two critical points have one negative real eigenvalue and two eigenvalues with positive real part, gives the unstable solution.

(viii) Boundedness of solutions in the Lorenz system:
There is a solid ellipsoid E given by

$$rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq c < \infty$$

such that all solutions of the Lorenz system enter E within finite time and therefore remain in E.
To prove it we take

$$\varphi(x, y, z) \equiv r^2 + \sigma y^2 + \sigma(z - 2r)^2 = c.$$

We shall show that there exists $c = c_{cr}$ such that for all $c > c_{cr}$ the trajectory is directed toward to the ellipsoid E at any point on the E. We have

$$\begin{aligned} \hat{n}_\varphi \cdot \dot{\tilde{}} &= \frac{\nabla\varphi}{|\nabla\varphi|} \cdot \dot{\tilde{}} = \frac{1}{|\nabla\varphi|}\left[\frac{\partial\varphi}{\partial x}\dot{x} + \frac{\partial\varphi}{\partial y}\dot{y} + \frac{\partial\varphi}{\partial z}\dot{z}\right] \\ &= \frac{1}{|\nabla\varphi|}[2rx\sigma(y - x) + 2\sigma y(rx - y - xz) + 2\sigma(z - 2r)(xy - bz)] \\ &= -\frac{2\sigma}{|\nabla\varphi|}[rx^2 + y^2 + b(z - r)^2 - br^2] < 0 \end{aligned}$$

So, the trajectory is directed inward to E if (x, y, z) lies inside of the ellipsoid

$$D \equiv a^2 + y^2 + b(z - r)^2 = br^2.$$

Now, for the ellipsoid D we have

$$\frac{x^2}{(\sqrt{br})^2} + \frac{y^2}{(r\sqrt{b})^2} + \frac{(z - r)^2}{r^2} = 1,$$

whose center is $(0,0,r)$ and the length of the semi-axes are $\sqrt{b}r, r\sqrt{b}, r$ respectively. Similarly, the center of the ellipsoid $E \equiv r^2 + \sigma y^2 + \sigma(z - 2r)^2 = c$ is $(0,0,2r)$ and the length of semi-axes are $\sqrt{c/r}, \sqrt{c/\sigma}$, and $\sqrt{c/\sigma}$, respectively. Since the x and y coordinates of the centers of both the ellipsoids are 0,0 respectively, the extent of the ellipsoid E in the x and y directions exceed the extent of the ellipsoid D in the same direction if

$$\sqrt{c/r} > \sqrt{br}; \quad \sqrt{c/\sigma} > r\sqrt{b}$$

that is, $c > br^2; c > b\sigma r^2$.
Next, along the z-axis the ellipsoid D is contained

$$0 = r - r < z < r + r = 2r$$

while for the ellipsoid E,

$$2r - \sqrt{c/\sigma} < z < 2r + \sqrt{c/\sigma}.$$

But $2r < 2r + \sqrt{c/\sigma}$ for all c. So, the lowest point $(0,0,0)$ of the ellipsoid D lies above the lowest point $(0,0,2r - \sqrt{c/\sigma})$ of the ellipsoid E if$2r - \sqrt{c/\sigma} < 0$, that is, $c > 4r^2\sigma$.

Let $c = c_\sigma = \max\{br^2, b\sigma r^2, 4r^2\sigma\}$. Then the ellipsoid D lies entirely within the ellipsoid E_σ. Hence for any point (x, y, z) exterior to D, the trajectory is directed inward E. All such trajectories must enter E_{cr} after some finite time and remain inside as $\hat{n}_\varphi \cdot \dot{\underset{\sim}{r}}$ can never be positive (Fig. 6.21).

20.3: A Mapping of R into R as a Dynamical System:

We now proceed to consider the dynamical behaviour of the mapping. For examples, in mathematical biology, suppose the number of individuals of population with species at time t is $N_t (N_t \geq 0)$. After one unit of time this number is

$N_{t+1} = f(N_t)$ ………………………………..(20.23)

where f is determined by the birth and death process. We expect that $f(0) = 0, N_{t+1} > N_t$. If N_t is small then $N_{t+1} < N_t$.

If the number is large because natural bounds of the amount of available space and food. A simple model is given by the logistic equation

$N_{t+1} = N_t + rN_t - \frac{r}{k}N_t^2$ ………………………(20.24)

where r is the growth coefficient and k is the $+ve$ constant.

Let, $x_t = \frac{rN_t}{k(1+r)}$ $and\ a = (1 + r)$

Then the equations (20.24) becomes

$x_{t+1} = ax_t(1 - x_t)$ ….………………..(20.25)

We choose $x \in [0,1]$. If x_0 is given, then (20.25) gives the value of x_1 for $t = 1$; substituting again produces x_2 for $t = 2$ and so on.

Definition-1: Let M be the smooth manifold; the c'-mapping $\varphi: R \times M \to M$ is a dynamical system for all $x \in M$ if

(i) $\varphi(0, x) = x$

(ii) $\varphi(t, \varphi(t_0, x)) = \varphi(t + t_0, x)$.

For continuous system $t, t_0 \in R$ and for discrete system $t, t_0 \in Z$.

In the continuous system the mapping φ is called a flow on $M \subset R^n$ generated by the autonomous initial value problem $\dot{\vec{x}} = \vec{f}(\vec{x}), \vec{x}(t_0) = \vec{x}_0$ is also a dynamical system.

Returning to equation (20.23) we note that $f(N_t)$ may have the fixed point N_0 (say) so that $f(N_0) = N_0$ which is called a periodic solution on a periodic point of f. It is also possible that after applying the mapping k-times, we are returning in N_0 so that $f^k(N_0) = N_0$.

Definition-2: Let x_0 be the fixed point of f. Then x_0 is asymptotically stable if there exists a nbd U of such that $\lim_{n\to\infty} f^n(x) = x_0, for\ all\ x \in U$. The point x_0 is an asymptotically stable periodic point of f with periodic k if x_0 is an asymptotically stable fixed point of f^k.

Definition-3: The domain of attraction of an asymptotically stable point x_0 with period $k$is the set of points which converge to x_0 by iteration of the mapping. The set $\{f^k(x_0)\}_{k=0}^{\infty}$ is called the orbit of x_0.

Definition-4:A point x_0 is called a periodic point of the mapping f if the orbit of x_0 is bounded and $x_0, k \in N$ exists such that $\lim_{n\to\infty} f^{n+k}(x_0)$ exists. In this case f is called chaotic.

20.4 Strange Attractors:

For certain values of parameters$\sigma, r\ and\ b$ the system

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

has a strong attraction set. The attractor A of this system is made up of an infinite number of branched surfaces which are interleaved and which intersect, however the trajectories of this system in A do not intersect but move from one branched surface to another as they circulate through the apparent branch. The closed invariant set A contains

(i) A countable set of periodic orbits of arbitrarily large period
(ii) An uncountable set of non-periodic motions
(iii) A dense orbit, the attracting set A having the properties is referred to as a strange attractor.

20.5 Chaos:

In the development of science in the twentieth century, philosophers and scientists convinced that there could be a motion even for a simple system which is erratic in nature, not simply periodic or quasiperiodic. Moreover, the behaviors of the motion may be unpredictable and therefore long-range prediction is impossible. The science of unpredictability has immense interest. The debate on the cause of unpredictability is continuing over centuries. The great physicist Albert Einstein wrote a letter to Max Born regarding unpredictability in the cosmos. He wrote: "You believe in the God who plays dice, and I in complete law and order." In fact, nothing in the universe behaves in a way that is predictable totally forever. The perceptions of the infinite, infimum, and their connections with finite are a matter of great concern in science and philosophy. Even there is a order in unpredictable motions. But how order and chaos coexist. What are the laws underlying in chaotic motion? With the advancement of science and computing power it is believed that a simple deterministic system can have very complicated dynamics which is inherently present in the system itself. On the other hand, for an infinitesimal change in system's initial setup the dynamics as a whole may completely change. While studying the unpredictable behaviors of a system, the American mathematician James Alan Yorke had introduced the term 'Chaos' for random looking dynamics of simple deterministic systems. In Greek mythology, 'Chaos' is defined as an infinite formless structure. However, the precise definition of chaos either literarily or mathematically is lacking behind till date because of its multi-length scales motions with formless structures at infinitum. The appearance of chaotic motion has no definite routes. In this book we shall present a basic understanding of what chaos is and its mathematical theory under some assumptions. In mathematical framework chaos is a phenomenon exhibiting "sensitive dependence on infinitesimally different initial set-ups" and topologically "mixing". The chaotic orbits are generally aperiodic and named as "strange" by Rulle and Takens in 1971 that have fractional dimensions, a new discovery in the twentieth century's nonlinear science. There are connections with chaos and fractal objects. Nonlinearity and dimensionality (≥ 3) are the key requirements for chaos in continuous systems while in discrete systems, even a one-dimensional linear system may exhibit chaotic motion provided the system has lack of differentiability. Chaotic phenomena abound in Nature and in manmade devices.

The perception of unpredictable behavior in deterministic systems had been conceived and reported in the works of theFrench mathematician Henri Poincare. George David Birkhoff, A.M. Lyapunov, M.L. Cartwright, J.E. Littlewood, Andrey Kolmogorov, Stephen Smale, and coresearchers were noteworthy workers at the early stage of developments in chaos theory, particularly for mathematical foundation of chaos. The chaotic motion in atmospheric flows was first described by the American meteorologist, Edward Lorenz in the year 1963 from numerical experiments on convective patterns for a very simplified model. He found that the solutions never settled down to fixed points or periodic orbits of this simple system. Trajectories oscillate in an irregular, nonperiodic pattern with completely different behaviors for infinitesimal small change of initial conditions. The solution structure when plotted in three-dimensional Eucledian space resembles as a surface of two wings of a butterfly. Lorenz pointed out that the solution set contained an infinite number of sheets, known as strange attractor with fractional dimension. The sensitive dependence of dynamical evolution for infinitesimal change of initial conditions is called the butterfly effect. In 1972, Lorenz talked on the butterfly effect in the American Association for the Scientific Progress and questioned: "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" He claimed that it was difficult to predict the long-range climate conditions correctly. The unpredictability is inherently present in the atmospheric flow itself. Therefore, the long-range prediction would be uncertain.

Quantifying chaos is a central issue for understanding chaotic phenomena. Experimental evidence and theoretical studies predict some qualitative and quantitative measures for quantifying chaos. In this chapter we discuss some measures such as universal sequence (U-sequence), Lyapunov exponent, renormalization group theory, invariant measure, etc., for quantifying chaotic motions. On the other hand, there are some universal numbers applicable for particular class of systems, for example, the Feigenbaum number, Golden mean, etc. The universality is an important feature in chaotic dynamics.

Chaotic nonlinear dynamics is a rapidly expanding field and has now been proved to have potential applications in many manmade devices, social sciences, chemical and biological processes, and computer science. The unexpected fluctuations in sudden occurrence of diseases may be explained with the help of chaos theory. Chaos theory is much helpful in designing true economic and monetary modeling in resource distribution, financial and policy-making decisions. Chaos synchronization theory has been used nowadays for sending secret messages and also in other areas.

20.5 Mathematical Theory of Chaos:

Chaos is ubiquitous. Chaotic motions are unpredictable. Philosophers and scientists are trying to understand logically how unpredictability occurs. How it can be expressed in mathematical setup. The unpredictability in chaos and its mathematical foundation are still not well established. The simple looking phenomenon such as the smoke column rising in still air from a cigarette, the oscillations and their patterns in the smoke column are so complicated to defy understanding. Similarly, the weather forecasting and the world stock market prices are the systems that fluctuate with time in a random, irregular ways that the long-term predictions do not often match with reality. Chaos is a deterministically unpredictable phenomenon. In the evolution of chaotic orbit there are trajectories which do not settle down to fixed points or periodic orbits or quasiperiodic orbits as time tends to infinity. Even a deterministic system has no random or noisy inputs; an irregular behaviour may appear due to presence of nonlinearity, dimensionality, or non-differentiability of the system. Although the time evolution obeys strict deterministic laws, the system seems to behave according to its own free will. The mathematical definition of chaos introduces two notions, viz., the topological transitive property implying the mixing and the metrical property measuring the distance. Chaotic orbit may be expressed by fractals. Before defining chaos under the mathematical framework we discuss some preliminary concepts and definitions of topological and metric spaces which are essential for chaos theory.

Exercises:

1. State the characteristics of Lorentz equations.
2. Write down the short note of i) Chaos and ii)Strange attractors .

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